PROJECTION COMPLEXES, REVISITED: A SAMPLE

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1. Introduction

This is a shorter version of [BBFS], and its goal is to present the main
ideas and techniques from that paper. Among the results from [BBFS], we
only include the fact that the projection complexes are quasi-trees, and the
perturbation of the projection distances.

2. Axioms

Let $\mathbf{Y}$ be a set and for each $Y \in \mathbf{Y}$ assume that we have a function

$$d_Y \colon \mathbf{Y}^2 \backslash \{(Y, Y)\} \to [0, \infty]$$

such that the following *strong projection axioms* are satisfied for some $\theta \geq 0$:

(SP 1) $d_Y(X, Z) = d_Y(Z, X)$;
(SP 2) $d_Y(X, Z) + d_Y(Z, W) \geq d_Y(X, W)$;
(SP 3) if $d_Y(X, Z) > \theta$ then $d_Z(X, W) = d_Z(Y, W)$ for all $W \in \mathbf{Y} \backslash \{Z\}$;
(SP 4) $d_Y(X, X) \leq \theta$;
(SP 5) $\#\{Y | d_Y(X, Z) > \theta\}$ is finite for all $X, Z \in \mathbf{Y}$.

The constant θ is the *projection constant*. Note that we allow $d_Y(X, Z) = \infty$.

The most important axiom is arguably (SP 3), which is a version of the
Behrstock inequality [Beh06]. As in [BBF15], we will use it to order certain
subsets of $\mathbf{Y}$, the idea being that if $d_Y(X, Z)$ is large, then Y is between X
and Z. We note that (SP 3) is in fact a more precise version of the Behrstock
inequality because the conclusion is an actual equality, not an approximate
one. This allows us to know the exact value of certain d_Y, and it is the key
to our much simpler proofs, compared to [BBF15].

Lemma 2.1. *(SP 3) and (SP 4) imply*

$$\min\{d_Y(X, Z), d_Z(X, Y)\} \leq \theta$$

Proof. If $d_Y(X, Z) > \theta$ then letting $W = Y$ in (SP 3) we have $d_Z(X, Y) = d_Z(Y, Y) \leq \theta$ by (SP 4). $\square$

Define $\mathbf{Y}_K(X, Z)$ to be the collection of $Y \in \mathbf{Y} \backslash \{X, Z\}$ such that $d_Y(X, Z) > K$.

Lemma 2.2 and Proposition 2.3 below say that, for large enough K,
$\mathbf{Y}_K(X, Z)$ can be totally ordered using the idea, as mentioned above, that

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Date: November 21, 2017.

if $d_Y(X, Z)$ is large then Y is between X and Z. The order has several equivalent characterizations, which is good for applications, and they are listed in Lemma 2.2:

Lemma 2.2. *For $Y_0, Y_1 \in \mathbf{Y}_{2\theta}(X, Z)$ the following conditions are equivalent:*

(1) $d_{Y_0}(X, Y_1) > \theta$;
(2) $d_{Y_1}(Y_0, W) = d_{Y_1}(X, W)$ for all $W \neq Y_1$;
(3) $d_{Y_1}(X, Y_0) \leq \theta$;
(4) $d_{Y_1}(Y_0, Z) > \theta$;
(5) $d_{Y_0}(W, Y_1) = d_{Y_0}(W, Z)$ for all $W \neq Y_0$;
(6) $d_{Y_0}(Y_1, Z) \leq \theta$.

Proof. By Lemma 2.1, $(1) \Rightarrow (3)$ and $(4) \Rightarrow (6)$. By (SP 2), $(3) \Rightarrow (4)$ and $(6) \Rightarrow (1)$. By (SP 3), $(1) \Rightarrow (2)$ and $(4) \Rightarrow (5)$. Since $Y_1 \in \mathbf{Y}_{2\theta}(X, Z)$ by letting $W = Z$ we have $(2) \Rightarrow (4)$ and similarly $(5) \Rightarrow (1)$. □

Given $Y_0, Y_1 \in \mathbf{Y}_{2\theta}(X, Z)$ we define $Y_0 < Y_1$ if any one of $(1) - (6)$ hold.

Proposition 2.3. *The relation $<$ defines a total order on $\mathbf{Y}_{2\theta}(X, Z)$ that extends to a total order on $\mathbf{Y}_{2\theta}(X, Z) \cup \{X, Z\}$ with least element X and greatest element Z. Furthermore if $Y_0 < Y_1 < Y_2$ then $d_{Y_1}(Y_0, Y_2) = d_{Y_1}(X, Z)$.*

Notice that with a coarse version of (SP 3) there would be no hope to obtain the last conclusion as stated.

Proof. By swapping Y_0 and Y_1 we see that $Y_0 < Y_1$ if and only if $Y_1 \not< Y_0$. So any two elements of $\mathbf{Y}_{2\theta}(X, Z)$ can be compared.

Now we check transitivity of the order. If $Y_0 < Y_1$ and $Y_1 < Y_2$ we apply (2) for $Y_0 < Y_1$ with $W = Y_2$ and then again to $Y_1 < Y_2$ with $W = Z$ to see that $d_{Y_1}(Y_0, Y_2) = d_{Y_1}(X, Y_2) = d_{Y_1}(X, Z) > 2\theta$. Applying (SP 3) and then (5) we have $d_{Y_2}(Y_0, Z) = d_{Y_2}(Y_1, Z) = d_{Y_2}(X, Z) > 2\theta$. Therefore $Y_0 < Y_2$ and the total order is well defined on $\mathbf{Y}_{2\theta}(X, Z)$. We can extend it to a total order on $\mathbf{Y}_{2\theta}(X, Z) \cup \{X, Z\}$ by declaring X to be the least element and Z the greatest element.

Observe that we have also shown that $d_{Y_1}(Y_0, Y_2) = d_{Y_1}(X, Z)$ if $Y_0 < Y_1 < Y_2$. □

Lemma 2.4. *Let $K \geq 2\theta$. If $Y_0, Y_1 \in \mathbf{Y}_K(X, Z) \cup \{X, Z\}$ and $d_Y(Y_0, Y_1) > K$ then $Y \in \mathbf{Y}_K(X, Z)$.*

Proof. We assume $Y_i \notin \{X, Z\}$, since in those cases the proof is similar and easier.

We can assume that $Y_0 < Y_1$. By (SP 3) $d_{Y_0}(X, Y) = d_{Y_0}(X, Y_1)$, and the latter is $> \theta$ by Lemma 2.2-(1). From the equivalence of (1) and (2) we get $K < d_Y(Y_0, Y_1) = d_Y(X, Y_1)$.

Using (SP3) again we get $d_{Y_1}(Y, Z) = d_{Y_1}(X, Z)$, and the latter is $> K$ by assumption. Hence, again by (SP3), $d_Y(X, Z) = d_Y(X, Y_1) = d_Y(Y_0, Y_1) > K$, as required. □

3. The projection complex

Fix $K \geq 2\theta$ and define the graph $\mathcal{P}_K(\mathbf{Y})$ with vertex set $\mathbf{Y}$ and an edge between any two vertices X and Z with $\mathbf{Y}_K(X, Z) = \emptyset$. We denote the distance in $\mathcal{P}_K(\mathbf{Y})$ simply by d, even though it depends on K.

We first note that $\mathcal{P}_K(\mathbf{Y})$ is connected.

Lemma 3.1. *If $K \geq 2\theta$ and $X, Z \in \mathbf{Y}$ then $\mathbf{Y}_K(X, Z) \cup \{X, Z\} = \{X < X_1 < \cdots < X_k < Z\}$ is a path in $\mathcal{P}_K(\mathbf{Y})$.*

Proof. By Lemma 2.4, if Y is the immediate predecessor of Y' in the total order on $\mathbf{Y}_K(X, Z) \cup \{X, Z\}$ then $\mathbf{Y}_K(Y, Y') = \emptyset$ and therefore $d(Y, Y') = 1$. □

The following lemma says that, when moving outside the ball of radius 2 around a vertex Z of $\mathcal{P}_K(\mathbf{Y})$, the projection to Z varies slowly, where slowly is independent of K.

Lemma 3.2. *If $K \geq 2\theta$ then the following holds. Let $X_0, X_1, Z \in \mathbf{Y}$ with $d(X_0, X_1) = 1$ and $d(X_0, Z) \geq 2$. Then $d_Z(X_0, X_1) \leq \theta$.*

Proof. Since $d(X_0, Z) \geq 2$ there exists a $Y \in \mathbf{Y}_K(X_0, Z)$ and therefore by (SP 3) $d_Z(X_0, X_1) = d_Z(Y, X_1)$. If $d_Z(Y, X_1) = d_Z(X_0, X_1) > \theta$ then by (SP 3) $d_Y(X_0, X_1) = d_Y(X_0, Z) > K$, a contradiction with $\mathbf{Y}_K(X_0, X_1) = \emptyset$. □

The following lemma and its corollary are the key to proving that $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree. They say that, when moving outside the ball of radius 3 around a vertex Z of $\mathcal{P}_K(\mathbf{Y})$, the projection to Z basically does not change.

Lemma 3.3. *If $K \geq 3\theta$ then the following holds. Let $X_0, \ldots, X_k$ be a path in $\mathcal{P}_K(\mathbf{Y})$ and $Z \in \mathbf{Y}$ with $d(X_i, Z) \geq 3$. Then greatest elements of $\mathbf{Y}_{3\theta}(X_i, Z)$ all agree.*

Proof. We can assume $k = 1$. Let Y_0 and Y_1 be the corresponding greatest elements and assume they are distinct. By Corollary 2.4, $\mathbf{Y}_{3\theta}(Y_i, Z) = \emptyset$ so $d(Y_i, Z) = 1$ and $d(X_i, Y_i) \geq 2$. Applying Lemma 3.2 we see that $d_{Y_i}(X_0, X_1) \leq \theta$ and therefore by (SP 2), $d_{Y_i}(X_{1-i}, Z) > 2\theta$. In particular, both Y_0 and Y_1 are in $\mathbf{Y}_{2\theta}(X_i, Z)$ for $i = 0, 1$. We can assume that $Y_0 < Y_1$ in $\mathbf{Y}_{2\theta}(X_0, Z)$. By Lemma 2.2(6) this means that $d_{Y_0}(Y_1, Z) \leq \theta$ and so we also have $Y_0 < Y_1$ in $\mathbf{Y}_{2\theta}(X_1, Z)$. In particular, $d_{Y_1}(X_0, Z) = d_{Y_1}(Y_0, Z) = d_{Y_1}(X_1, Z) > 3\theta$, contradicting the assumption that Y_0 is the greatest element of $\mathbf{Y}_{3\theta}(X_0, Z)$. □

Corollary 3.4. *If $K \geq 3\theta$ then the following holds. Let $X_0, \ldots, X_k$ be a path in $\mathcal{P}_K(\mathbf{Y})$ and $Z \in \mathbf{Y}$ with $d(X_i, Z) \geq 3$. Then $d_Z(X_i, X_j) \leq \theta$ for all i, j.*

Proof. By Lemma 3.3, there exists a $Y \in \mathbf{Y}$ that is the greatest element of all of the $\mathbf{Y}_{3\theta}(X_i, Z)$. We now have $d_Z(X_i, X_j) = d_Z(X_i, Y) \leq \theta$ by (SP 3) and Lemma 2.1. □

We can now use Manning's *bottleneck condition* [Man05] to show that $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree. We will use a variant of Manning's condition that is described in [BBF15]: Let X be a connected simplicial graph with its usual combinatorial metric and $D \geq 0$. Assume that for all vertices $v_0, v_1 \in X^{(0)}$ there is a path p such that the D-Hausdorff neighborhood of any path from v_0 to v_1 contains p. Then X is a quasi-tree.

Theorem 3.5. *For $K \geq 3\theta$, $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree.*

Proof. By Lemma 3.1 $\mathbf{Y}_K(X,Z) \cup \{X,Z\} = \{X < X_1 < \cdots < X_k < Z\}$ is a path in $\mathcal{P}_K(\mathbf{Y})$. Let $X = Y_0, Y_1, \ldots, Y_n = Z$ be an arbitrary path from X to Z. Since $d_{X_i}(X,Z) > K \geq 3\theta$ by Corollary 3.4 there must be a Y_j such that $d(Y_j, X_i) \leq 2$. Therefore $\mathcal{P}_K(\mathbf{Y})$ satisfies the bottleneck condition and is a quasi-tree. $\qquad\square$

4. Forcing sequences

Let $\mathbf{Y} = \{(Y, \rho_Y)\}$ be a collection of metric spaces and for each distinct $Y, Z \in \mathbf{Y}$ assume that we have sets $\pi_Y(Z) \subseteq Y$ and $\pi_Z(Y) \subseteq Z$. The π_Y are *projection maps*. Assume that for some $\theta \geq 0$ we have for any $X \neq Y$,

(P0') $\operatorname{diam}(\pi_X(Y)) \leq \theta$.

Denote by $d_Y^\pi(X, Z)$ the ρ_Y-diameter of $\pi_Y(X) \cup \pi_Y(Z)$.

Consider the following axioms from [BBF15]. For each pairwise distinct $X, Y, Z \in \mathbf{Y}$, for some $\theta \geq 0$:

(P0) $d_Y(X, X) \leq \theta$.
(P1) if $d_Y(X, Z) > \theta$ then $d_X(Y, Z) \leq \theta$.
(P2) $\{W \neq X, Z : d_W(X, Z) > \theta\}$ is finite.
(P3) $d_Y(X, Z) = d_Y(Z, X)$
(P4) $d_Y(X, Z) + d_Y(Z, W) \geq d_Y(X, W)$

The d_Y^π from projection maps π_Y satisfy (P3) and (P4). (P0') is equivalent to (P0). Families of metric spaces with projection maps satisfying (P1) and (P2) as well occur naturally in many contexts. See the introduction to [BBF15] for some examples.

The goal of this section is to prove the following theorem.

Theorem 4.1. *Assume that $\mathbf{Y} = \{(Y, \rho_Y)\}$ is a collection of metric spaces with $\{d_Y^\pi\}$ satisfying (P0) - (P4) with constant θ.*

Then there are $\{d_Y\}$ satisfying (SP 1) -(SP 5) for the constant 11θ such that

$$d_Y^\pi - 2\theta \leq d_Y \leq d_Y^\pi + 2\theta.$$

Mimicking the earlier section we let $\mathbf{Y}_K^\pi(X, Z)$ be the collection of $Y \in \mathbf{Y} \backslash \{X, Z\}$ such that $d_Y^\pi(X, Z) > K$.

4.1. **Modifying the distance** d^π**.** We assume d_Y^π satisfy (P0) - (P4).

The first step is to modify d^π to achieve monotonicity (see Lemma 4.4). Recall from [BBF15] that for $X \neq Z$ we define $\mathcal{H}(X, Z)$ as the set of pairs $(X', Z') \in \mathbf{Y} \times \mathbf{Y}$ such that one of the following holds.

- $d_X^\pi(X', Z'), d_Z^\pi(X', Z') > 2\theta$,
- $X = X'$ and $d_Z^\pi(X, Z') > 2\theta$,
- $Z = Z'$ and $d_X^\pi(X', Z) > 2\theta$,
- $X = X'$ and $Z = Z'$.

Lemma 4.2. *If $d_Y^\pi(X, Z) > 2\theta$ and $(X', Z') \in \mathcal{H}(X, Z)$ then $d_Y^\pi(X, X'), d_Y^\pi(Z, Z') \leq \theta$. In particular, $|d_Y^\pi(X, Z) - d_Y^\pi(X', Z')| \leq 2\theta$.*

Proof. By the triangle inequality (P4)

$$d_X^\pi(X', Y) + d_X^\pi(Y, Z') \geq d_X(X', Z') > 2\theta$$

and therefore $\max\{d_X^\pi(X', Y), d_X^\pi(Y, Z')\} > \theta$. After possibly permuting X' and Z' we can assume $d_X^\pi(X', Y) > \theta$ and therefore by (P1) we have $d_Y^\pi(X, X') \leq \theta$. By another application of the triangle inequality

$$d_Y^\pi(X, X') + d_Y^\pi(X', Z) \geq d_Y^\pi(X, Z) > 2\theta$$

and since $d_Y^\pi(X, X') \leq \theta$ this implies that $d_Y^\pi(X', Z) > \theta$. Therefore (P1) implies that $d_Z^\pi(X', Y) \leq \theta$. Now, replacing X with Z in the above application of the triangle inequality (P4) we have $\max\{d_Z^\pi(X', Y), d_Z^\pi(Y, Z')\} > \theta$ and therefore $d_Z^\pi(Y, Z') > \theta$ since we have just seen that the other term is $\leq \theta$. Another application of (P1) gives that $d_Y^\pi(Z, Z') \leq \theta$.

The final inequality follows from the triangle inequality (P4). $\square$

Corollary 4.3. *If $d_Y^\pi(X, Z) > 4\theta$ then $\mathcal{H}(X, Z) \subseteq \mathcal{H}(X, Y)$.*

Proof. Suppose $(X', Z') \in \mathcal{H}(X, Z)$. We will again assume the first bullet holds and leave the other cases to the reader. To show $(X', Z') \in \mathcal{H}(X, Y)$ it suffices to argue that $d_Y^\pi(X', Z') > 2\theta$, and this follows from $d_Y^\pi(X, Z) > 4\theta$ and the lemma. $\square$

We now define the modified distance

$$\tilde{d}_Y(X, Z) = \sup_{(X', Z') \in \mathcal{H}(X, Z)} d_Y^\pi(X, Z)$$

if $d_Y^\pi(X, Z) > 2\theta$, and $\tilde{d}_Y(X, Z) = 2\theta$ otherwise. Thus

$$d_Y^\pi(X, Z) \leq \tilde{d}_Y(X, Z) \leq d_Y^\pi(X, Z) + 2\theta$$

The triangle inequality for $\tilde{d}$ holds only up to an error of 2θ. What we gain with this modification is the following monotonicity property.

Lemma 4.4. *If $\tilde{d}_Y(X, Z) > 5\theta$ and $\tilde{d}_W(Y, Z) > 7\theta$ then $\tilde{d}_Y(X, W) \geq \tilde{d}_Y(X, Z)$.*

Proof. We have $d_W^\pi(Y, Z) > 5\theta$ so $d_Y^\pi(W, Z) \leq \theta$. Likewise, $d_Y^\pi(X, Z) > 3\theta$ so $d_Y^\pi(X, W) \geq d_Y^\pi(X, Z) - d_Y^\pi(W, Z) > 2\theta$ and so $d_W^\pi(X, Y) \leq \theta$. Thus $d_W^\pi(X, Z) \geq d_W^\pi(Y, Z) - d_W^\pi(X, Y) > 4\theta$. Corollary 4.3 gives $\mathcal{H}(X, W) \supseteq \mathcal{H}(X, Z)$. We saw above that $d_Y^\pi(X, W) > 2\theta$ and the statement follows. $\square$

4.2. **The second (and final) modification of d^π.** To prove the theorem we need to modify the d_Y^π so that they satisfy the projection axioms (SP 1)-(SP 5). The key notion to do so is that of a forcing sequence, which uses the first modification $\tilde{d}$.

Also, if d_Y^π are obtained from projections π_Y, this modification is realized by modifications of π_Y.

Definition 4.5. A *K-forcing sequence* is a sequence $Y = Y_0, \ldots, Y_n = Z$ of distinct elements of $\mathbf{Y}$ so that $\tilde{d}_{Y_i}(Y_{i-1}, Y_{i+1}) > K$ (for all $i = 1, \ldots, n-1$).

Notice that if $X \neq Z$ then $X = Y_0, Y_1 = Z$ is a (usually non-maximal) forcing sequence.

Lemma 4.6. *Let $Y_0, Y_1, \cdots, Y_n$ be a 4θ-forcing sequence. Then*

(i) $d_{Y_n}^\pi(Y_0, Y_{n-1}) \leq \theta$,
(ii) $|d_{Y_j}^\pi(Y_i, Y_k) - d_{Y_j}^\pi(Y_{j-1}, Y_{j+1})| \leq 2\theta$ *for all $i < j < k$.*

Proof. We prove (i) by induction on n, starting with the obvious case $n = 1$. Suppose that it is true for a given n and let us prove it for $n + 1$. Observe that $d_{Y_i}^\pi(Y_{i-1}, Y_{i+1}) > 2\theta$.

Since $d_{Y_n}^\pi(Y_0, Y_{n-1}) \leq \theta$, by the triangle inequality we have $d_{Y_n}^\pi(Y_0, Y_{n+1}) > \theta$, so that $d_{Y_{n+1}}^\pi(Y_0, Y_n) \leq \theta$, as required.

To prove (ii) note that both $Y_i, Y_{i+1}, \ldots, Y_j$ and $Y_k, Y_{k-1}, \ldots, Y_j$ are 4θ-forcing sequences. We apply (i) to each of them and (ii) then follows from the triangle inequality. $\square$

The lemma below tells us when we can insert elements in forcing sequences, and it will be used to show that if $\tilde{d}_W(X, Z)$ is large, then any maximal forcing sequence from X to Z goes through W. Its proof uses the monotonicity of $\tilde{d}$.

Lemma 4.7. *Let $Y_0, \ldots, Y_n$ be a K-forcing sequence with $K \geq 7\theta$ and $W \in \mathbf{Y}$ such that $\tilde{d}_W(Y_i, Y_{i+1}) \geq K$. Then $Y_0, \ldots, Y_i, W, Y_{i+1}, \ldots, Y_n$ is a K-forcing sequence.*

Proof. We need to argue that $\tilde{d}_{Y_i}(Y_{i-1}, W), \tilde{d}_{Y_{i+1}}(W, Y_{i+2}) > K$. Both follow from Lemma 4.4, e.g. $\tilde{d}_{Y_i}(Y_{i-1}, W) \geq \tilde{d}_{Y_i}(Y_{i-1}, Y_{i+1}) > K$. $\square$

Lemma 4.8. *For $K \geq 7\theta$, any K-forcing sequence from X to Z can be refined into a maximal one.*

Proof. The obvious process of refinement, using Lemma 4.7, must terminate by Lemma 4.6(ii) and (P2). $\square$

Lemma 4.9. *Let $Y_0, \ldots, Y_n$ be a maximal K-forcing sequence, $K \geq 7\theta$, and let $W \in \mathbf{Y}$ with $d_W^\pi(Y_0, Y_n) > K + 2\theta$. Then $W = Y_i$ for some $i \in \{1, \ldots, n-1\}$.*

Proof. We assume that W is distinct from all the Y_i and derive a contradiction.

By Lemma 4.6, $d_{Y_i}^\pi(Y_0, Y_n) > 2\theta$. We first observe that if $d_W^\pi(Y_i, Y_n) > \theta$ then $d_{Y_i}^\pi(Y_0, W) \geq d_{Y_i}^\pi(Y_0, Y_n) - d_{Y_i}^\pi(W, Y_n) > \theta$ since by (P1) $d_{Y_i}^\pi(W, Y_n) \leq \theta$. Again applying (P1) we have $d_W^\pi(Y_0, Y_i) \leq \theta$.

Let i be the smallest index from $1, \ldots, n-1$ such that $d_W^\pi(Y_i, Y_n) \leq \theta$, assuming such an i exists. Then $d_W^\pi(Y_{i-1}, Y_i) \geq d_W^\pi(Y_0, Y_n) - d_W^\pi(Y_0, Y_{i-1}) - d_W^\pi(Y_i, Y_n) > K$ since $d_W^\pi(Y_0, Y_{i-1}) \leq \theta$ by the previous paragraph and $d_W^\pi(Y_i, Y_n) \leq \theta$ by assumption. (If $i = 1$ then there is no middle term on the right hand side of the inequality.) If no such i exists we let $i = n$ and see that $d_W^\pi(Y_{i-1}, Y_i) = d_W^\pi(Y_{n-1}, Y_n) \geq d_W^\pi(Y_0, Y_n) - d_W^\pi(Y_0, Y_{n-1}) > K + \theta$.

In all cases we have found an i such that $\tilde{d}_W(Y_{i-1}, Y_i) \geq d_W^\pi(Y_{i-1}, Y_i) > K$ and therefore $Y_0, \ldots, Y_{i-1}, W, Y_i, \ldots, Y_n$ is a K-forcing sequence by Lemma 4.7, contradicting our maximality assumption. $\qquad\square$

Lemma 4.10. *Let $Y_0, \ldots, Y_{n-1}, Y_n$ be a maximal K-forcing sequence with $K \geq 7\theta$ and suppose that $X \in \mathbf{Y}$ satisfies $d_{Y_0}^\pi(X, Y_n) > K + \theta$. Then there exists a maximal K-forcing sequence from X to Y_n with penultimate element Y_{n-1}.*

Proof. By Lemma 4.6, $d_{Y_0}^\pi(Y_1, Y_n) \leq \theta$ so

$$\tilde{d}_{Y_0}(X, Y_1) \geq d_{Y_0}^\pi(X, Y_1) \geq d_{Y_0}^\pi(X, Y_n) - d_{Y_0}^\pi(Y_1, Y_n) > K.$$

Therefore $X, Y_0, \ldots, Y_n$ is a K-forcing sequence. Any maximal refinement will have the required property for if in the refinement an element appeared between Y_{n-1} and Y_n then the original sequence would not be maximal. $\quad\square$

Definition 4.11 (Penultimate elements). For distinct elements $X, Z \in \mathbf{Y}$ define a subset $P_Z(X) = \{W\} \subset \mathbf{Y}$, where W are all penultimate elements of maximal 7θ-forcing sequences from X to Z. Note that $P_Z(X)$ is not empty.

When $Y \neq X, Z$, we define

$$d_Y(X, Z) = \sup d_Y^\pi(W_1, W_2),$$

where $W_1 \in P_Y(X), W_2 \in P_Y(Z)$.

Lemma 4.12. *We have*

$$d_Y^\pi - 2\theta \leq d_Y \leq d_Y^\pi + 2\theta.$$

Proof. By Lemma 4.6 if W is the penultimate element of a 7θ-forcing sequence from X to Z then $d_Z^\pi(X, W) \leq \theta$. The inequalities follow from triangle inequality of d_Y^π. The second claim is clear. $\qquad\square$

Lemma 4.13. *If $d_Y(X, Z) > 11\theta$ then $P_Z(X) = P_Z(Y)$.*

Proof. By Lemma 4.12 if $d_Y(X, Z) > 11\theta$ then $d_Y^\pi(X, Z) > 9\theta$. By Lemma 4.9 if $X = Y_0, \ldots, Y_n = Z$ is a maximal 7θ-forcing sequence then $Y = Y_i$ for some $i \in \{1, \ldots, n-1\}$. Then $Y_i, \ldots, Y_n$ is a maximal 7θ-forcing sequence from Y to Z and it follows that $P_Z(Y) \supseteq P_Z(X)$.

By Lemma 4.10 any maximal 7θ-forcing sequence from Y to Z can be extended to a maximal 7θ-forcing sequence from X to Z with the same penultimate element so $P_Z(X) \supseteq P_Z(Y)$.

We showed $P_Z(X) = P_Z(Y)$. $\hfill\square$

Remark 4.14. Set $\pi'_Y(X) = \cup \pi_Y(W)$, where $W \in P_Y(X)$. One can say we perturbed the projection from $\pi_Y(X)$ to $\pi'_Y(X)$, by at most θ in the Hausdorff distance, which induces the new distance d_Y. In Lemma 4.13, we have $\pi'_Z(X) = \pi'_Z(Y)$.

Proposition 4.15. *If d^π_Y satisfy (P0)-(P4), then d_Y satisfy the projection axioms (SP 1)-(SP 5) with projection constant 11θ.*

Proof. (SP 1) and (SP 2) are trivial. (SP 3) is exactly Lemma 4.13 and (SP 5) follows from (P2) and Lemma 4.12. The other axioms are clear. $\hfill\square$

Lemma 4.12 and Proposition 4.15 complete the proof of Theorem 4.1.

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