Week 1 exercises

1. View the 2-torus T^2 as $\mathbb{R}^2/\mathbb{Z}^2$ and consider $-I : T^2 \to T^2$. Show that it has 4 fixed points and the quotient is homeomorphic to S^2. We will view the images of the fixed points as punctures on the quotient S^2, so we will regard the quotient as $\Sigma_{0,4,0}$. Prove that $Mod(\Sigma_{0,4,0})$ is isomorphic to the subgroup of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$ consisting of elements congruent to I mod 2. Hints: First construct a homomorphism $Mod(\Sigma_{0,4,0}) \to PSL_2(\mathbb{Z})$ by lifting to T^2, so you should show that every homeomorphism fixing the punctures lifts. To show this homomorphism is 1-1, the key is to argue that the map $T^2 \to \Sigma_{0,4,0}$ induces a bijection between nontrivial isotopy classes of simple closed curves in T^2 and in $\Sigma_{0,4,0}$.

2. Suppose Σ is a hyperbolic surface with cusps and γ is a simple closed curve on Σ such that a deck transformation that preserves a lift $\tilde{\gamma}$ is parabolic. Show that γ bounds a punctured disk.

3. If α and β are simple closed curves that intersect transversely and no complementary component is a bigon, then they intersect minimally. We proved this in class assuming all complementary components are disks. Prove this statement in general. Hint: The key case is when a complementary region is a punctured bigon. The square tiling of the surface constructed in class will have a missing vertex. What does the metric completion of the universal cover look like? Metric completion of the α-cover? Set the argument from class in this completion.

4. Let α and β be two disjoint separating nontrivial nonhomotopic simple closed curves on Σ, so there are 3 complementary components, one of them, say R, bounded by $\alpha \cup \beta$. Let $h = T_\alpha T_\beta^{-1}$ and let $\tilde{h}$ be the lift of h to the universal cover $\mathbb{H}^2$ which is identity on a region that covers R. Describe the dynamics of the extension of $\tilde{h}$ to S^1_∞.

5. Prove the following relations among Dehn twists.

   (i) If f is any mapping class then $T_{f(a)} = fT_a f^{-1}$.

   (ii) If a, b are disjoint then $T_a T_b = T_b T_a$.

   (iii) If a, b intersect once transversally then $T_b(a) = T_a^{-1}(b)$ (draw the picture of both sides). Deduce, using (i) with $f = T_a T_b$, that $T_a T_b T_a = T_b T_a T_b$. This is the *braid relation.*

There are several other famous relations: root of Dehn twist, lantern relation, star relation. You may want to look them up.

6. Let X be a space, $x_0 \in X$ a basepoint and $f_t : X \to X$ a homotopy such that $f_0 = f_1 = id$ and $t \mapsto f_t(x_0)$ is a loop describing $\gamma \in \pi_1(X, x_0)$. Show that γ is in the center of $\pi_1(X, x_0)$. Deduce that the point pushing map $\pi_1(S, x_0) \to Mod(S, x_0)$ is injective provided the center of $\pi_1(S)$ is trivial.

7. If two vertices a, b in the complex $N(\Sigma)$ descibed in the class (nonseparating curves, intersection number 1) have intersection number M, show that they are connected by a path in $N(\Sigma)$ of length $\leq 10M + 10$. Can you find a better estimate?

8. Let Σ be a closed surface of genus $g \geq 2$ and let $N(\Sigma)$ and $X(\Sigma)$ (all curves, intersection number ≤ 1) be the two complexes from the class. Show that inclusion $N(\Sigma) \hookrightarrow X(\Sigma)$ is a quasi-isometry.

9. View $A \in SL_2(\mathbb{Z})$ as a diffeomorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$ and assume that $|TrA| > 2$, so A is Anosov. Prove that the periodic points of A are dense in the torus. Hint: Look at rational points.