Homework

These questions are intended to get the audience into the spirit of the course and to give me an idea of the background.

We denote by S_g the closed orientable surface of genus g. A simple closed curve in S_g is *essential* if it is not nullhomotopic; equivalently, it does not bound a disk.

1. Prove the equivalence of the two definitions of essential.
2. Prove that two disjoint simple closed curves in S_g are homotopic if and only if they cobound an annulus. If this happens we say they are *parallel*.
3. Suppose $S_g \to S_h$ is a d-fold covering map. Compute g as a function of h and d.
4. What is the largest number of pairwise disjoint, essential, nonparallel simple closed curves in S_g?
5. What is the largest number of pairwise disjoint, nonseparating, nonparallel simple closed curves in S_g?
6. What is the largest number of pairwise disjoint simple closed curves in S_g representing a linearly independent set in $H_1(S_g)$?
7. Suppose $\phi : \pi_1(S_g) \to F_k$ is a surjective homomorphism onto a free group F_k of rank k. Show that $k \leq g$.

   Hint 1: ϕ is induced by $f : S_g \to R_k$ where R_k is the wedge of k circles. Show that $H^1(f) : H^1(R_k) \to H^1(S_g)$ is an embedding onto a subspace where cup products are all 0 (Lagrangian subspace) and bound the dimension of a Lagrangian subspace.

   Hint 2: Make f transverse to the midpoints of edges, study the preimages and apply the previous problem.
8. For any two nonseparating simple closed curves in S_g there is a homeomorphism of S_g that takes one to the other.
9. Show that there is a number $\delta > 0$ with the following property. For any geodesic triangle ABC in the hyperbolic plane $\mathbb{H}^2$ and for any $p \in AB$ there is $q \in AC \cup BC$ with $d(p, q) < \delta$.

10. Show that there is a number $C > 0$ with the following property. Let ℓ be a geodesic line in $\mathbb{H}^2$ and $B \subset \mathbb{H}^2$ a metric ball disjoint from ℓ. Then the image of B under the nearest point projection $\mathbb{H}^2 \to \ell$ is an interval of length $< C$.

11. Show that δ and C of the previous two problems do not exist if hyperbolic plane is replaced by Euclidean plane.

12. Can you find the best possible δ and C in Problems 9 and 10?

13. Let f and g be two isometries of $\mathbb{H}^2$, f is hyperbolic, g is parabolic, and they both fix the same point at infinity. Show that the subgroup of the isometry group $Isom(\mathbb{H}^2)$ generated by f and g is not discrete.

14. Suppose that f and g are isometries of $\mathbb{H}^2$ and $gf = fg^2$. Show that g cannot be a hyperbolic (loxodromic) isometry.

15. Let H be a hexagon in $\mathbb{H}^2$ with all angles $\pi/2$. What is the area of H? Show that the group of isometries generated by reflections in the sides of H is discrete and that H is the fundamental domain for it.