Homework

These questions are intended to get the audience into the spirit of the course and to give me an idea of the background.

We denote by $S_g$ the closed orientable surface of genus $g$. A simple closed curve in $S_g$ is essential if it is not nullhomotopic; equivalently, it does not bound a disk.

1. Prove the equivalence of the two definitions of essential.

2. Prove that two disjoint simple closed curves in $S_g$ are homotopic if and only if they cobound an annulus. If this happens we say they are parallel.

3. Suppose $S_g \to S_h$ is a $d$-fold covering map. Compute $g$ as a function of $h$ and $d$.

4. What is the largest number of pairwise disjoint, essential, nonparallel simple closed curves in $S_g$?

5. What is the largest number of pairwise disjoint, nonseparating, nonparallel simple closed curves in $S_g$?

6. What is the largest number of pairwise disjoint simple closed curves in $S_g$ representing a linearly independent set in $H_1(S_g)$?

7. Suppose $\phi : \pi_1(S_g) \to F_k$ is a surjective homomorphism onto a free group $F_k$ of rank $k$. Show that $k \leq g$.
   
   Hint 1: $\phi$ is induced by $f : S_g \to R_k$ where $R_k$ is the wedge of $k$ circles. Show that $H^1(f) : H^1(R_k) \to H^1(S_g)$ is an embedding onto a subspace where cup products are all 0 (Lagrangian subspace) and bound the dimension of a Lagrangian subspace.
   
   Hint 2: Make $f$ transverse to the midpoints of edges, study the preimages and apply the previous problem.

8. For any two nonseparating simple closed curves in $S_g$ there is a homeomorphism of $S_g$ that takes one to the other.

9. Show that there is a number $\delta > 0$ with the following property. For any geodesic triangle $ABC$ in the hyperbolic plane $\mathbb{H}^2$ and for any $p \in AB$ there is $q \in AC \cup BC$ with $d(p, q) < \delta$. 

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10. Show that there is a number $C > 0$ with the following property. Let $\ell$ be a geodesic line in $\mathbb{H}^2$ and $B \subset \mathbb{H}^2$ a metric ball disjoint from $\ell$. Then the image of $B$ under the nearest point projection $\mathbb{H}^2 \to \ell$ is an interval of length $< C$.

11. Show that $\delta$ and $C$ of the previous two problems do not exist if hyperbolic plane is replaced by Euclidean plane.

12. Can you find the best possible $\delta$ and $C$ in Problems 9 and 10?

13. Let $f$ and $g$ be two isometries of $\mathbb{H}^2$, $f$ is hyperbolic, $g$ is parabolic, and they both fix the same point at infinity. Show that the subgroup of the isometry group $\text{Isom}(\mathbb{H}^2)$ generated by $f$ and $g$ is not discrete.

14. Suppose that $f$ and $g$ are isometries of $\mathbb{H}^2$ and $gf = f g^2$. Show that $g$ cannot be a hyperbolic (loxodromic) isometry.

15. Let $H$ be a hexagon in $\mathbb{H}^2$ with all angles $\pi/2$. What is the area of $H$? Show that the group of isometries generated by reflections in the sides of $H$ is discrete and that $H$ is the fundamental domain for it.