Knots: a handout for Summer Math Program

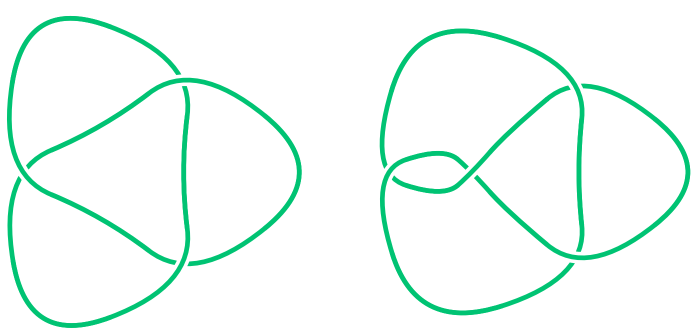
Mladen Bestvina

June 2007

1 Knots

Informally, a *knot* is a knotted loop of string. You can create one easily enough in one of the following ways:

- Take an extension cord, tie a knot in it, and then plug one end into the other.
- Let your cat play with a ball of yarn for a while. Then find the two ends (good luck!) and tie them together. This is usually a very complicated knot.
- Draw a diagram such as those pictured below. Such a diagram is a called a *knot diagram* or a *knot projection*.



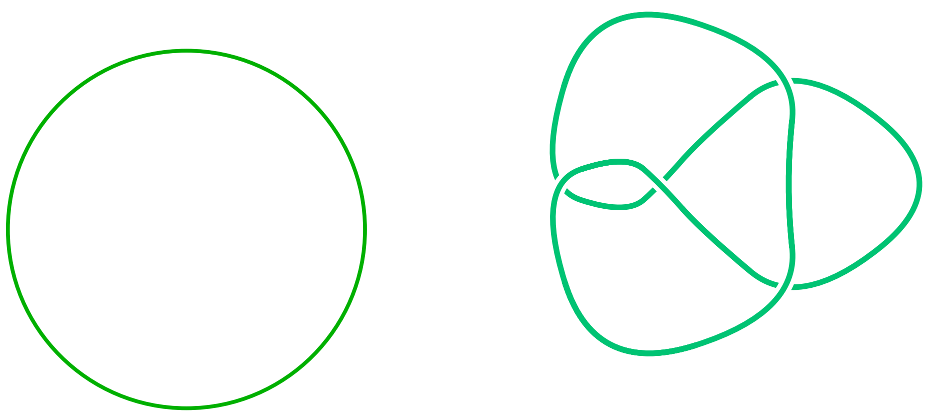
Trefoil and the figure 8 knot

The above two knots are the world's simplest knots. At the end of the handout you can see many more pictures of knots (from Robert Scharein's web site). The same picture contains many links as well. A *link* consists of several loops of string. Some links are so famous that they have names. For example, 2^2_1 is the *Hopf link*, 5^2_1 is the *Whitehead link*, and 6^3_2 are the *Borromean rings*. They have the feature that individual strings (or *components* in mathematical parlance) are untangled (or *unknotted*) but you can't pull the strings apart without cutting.

A bit of terminology: A *crossing* is a place where the knot crosses itself. The first number in knot's "name" is the number of crossings. Can you figure out the meaning of the other number(s)?

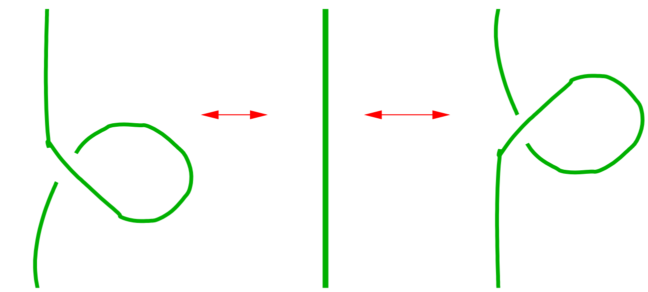
2 Reidemeister moves

There are many knot diagrams representing the same knot. For example, both diagrams below represent the *unknot*.

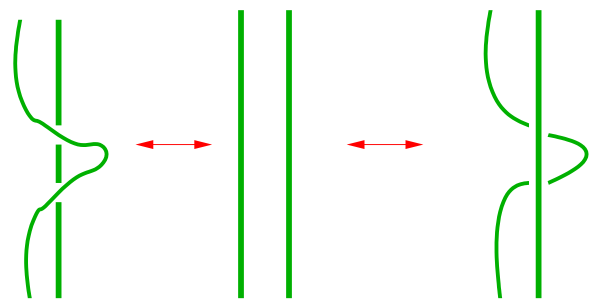


Two projections of the unknot

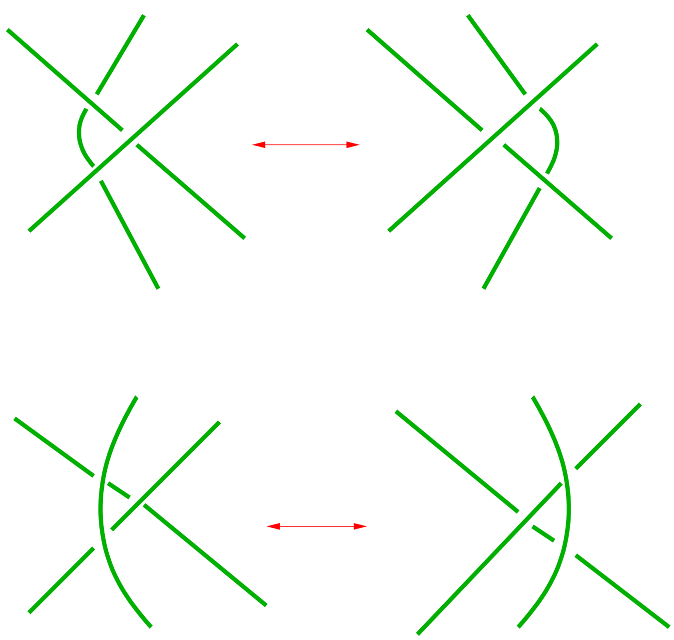
In fact, convince yourself that any of the following moves on a knot diagram will not change the knot it represents.



Type I Reidemeister move



Type II Reidemeister move



Type III Reidemeister move

1. Start with the knot diagram for the trefoil and change one of the crossings (i.e. make the upper strand go under the other one). Show that this new knot is an unknot by finding a sequence of Reidemeister moves that transforms it to a round circle.

2. Do the same with the figure 8 knot diagram.

3. Start with a round circle and then perform 4 Reidemeister moves in order to make the knot diagram more complicated. Then hand the picture to a friend to untangle it by finding simplifying Reidemeister moves.

There is a famous theorem proved by the German mathematician Kurt Reidemeister in 1920's that says:

If two knot diagrams represent the same knot (or a link) then one can be transformed to the other by a sequence of Reidemeister moves.

The catch here is that we don't know in advance how many moves will be needed.

4. The *mirror image* of a knot is obtained by reversing all crossings. Show that the mirror image of the figure 8 knot is the same knot as the figure 8 knot. It takes 9 Reidemeister moves to see this. You may want to experiment with a piece of string first.

It turns out that the mirror image of the trefoil is different from the trefoil.

Knot theory tries to answer questions such as:

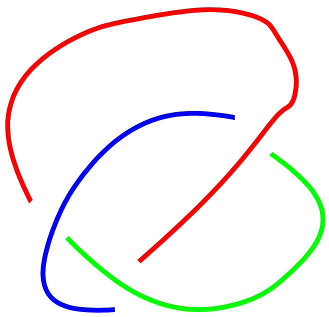
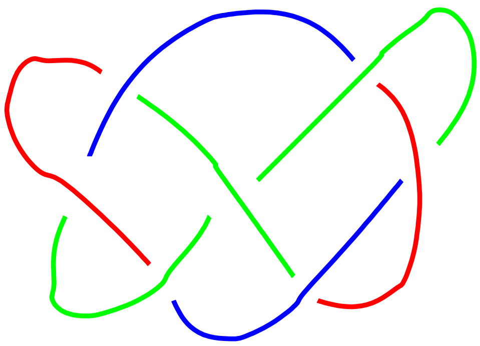
How to tell knots apart? How can we be sure that say the trefoil is really knotted and that there is no sequence of Reidemeister moves that transforms it to the unknot? Are the trefoil and the figure 8 knot really different? Can you pull Borromean rings apart without breaking them?

3 Tricolorability

A *strand* in a knot diagram is a continuous piece that goes from one under-crossing to the next. The number of strands is the same as the number of crossings.

A knot (or a link) is *tricolorable* if each strand can be colored in one of three colors with the following rules:

- At least two colors are used.
- At each crossing, either all three colors are present or only one color is present.

Trefoil and 7_4 are tricolorable

5. Decide which of the following are tricolorable: unknot, figure 8 knot, 2-component unlink, Hopf link, Whitehead link, 3-component unlink, Borromean rings. Which knots with 5, 6, 7 crossings?

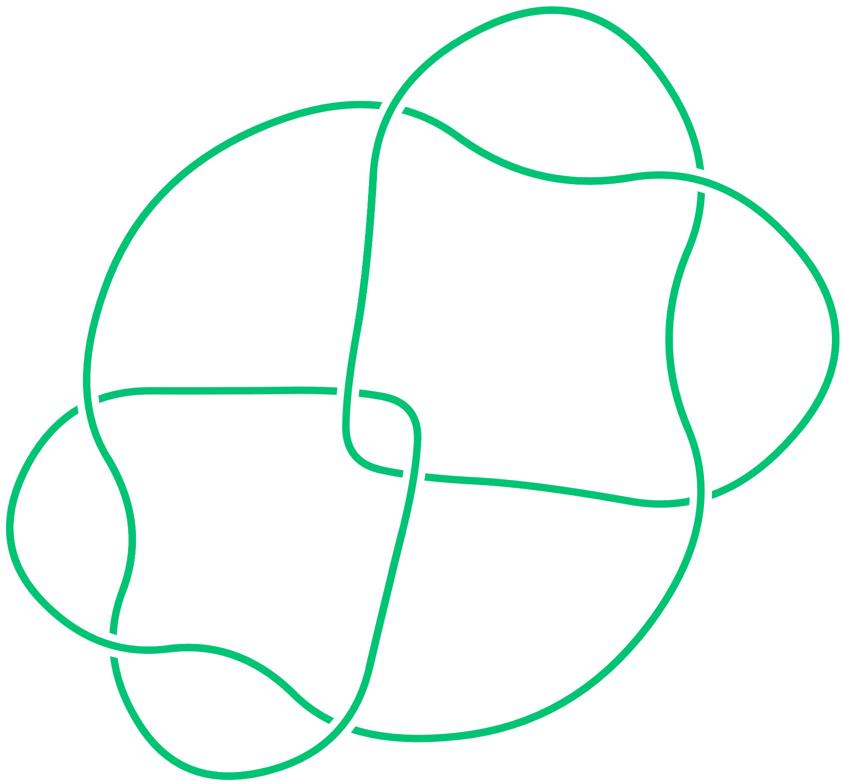
6. Show that if you start with a tricolorable knot diagram and you perform a Reidemeister move, the new knot diagram is also tricolorable.

Conclude the following:

Some knots are tricolorable and some are not, but to find out it is enough to check a single knot diagram for this knot.

7. Show that the trefoil is really a knot. Also show that the figure 8 knot is different from the trefoil, that the Hopf and Whitehead links cannot be pulled apart, and that Borromean rings cannot be pulled apart.

8. Show that "True Lover's Knot" is tricolorable. Can you use tricolorability to tell this knot apart from "False Lover's Knot" (reverse the two middle crossings).



True lover's knot

4 n-colorability

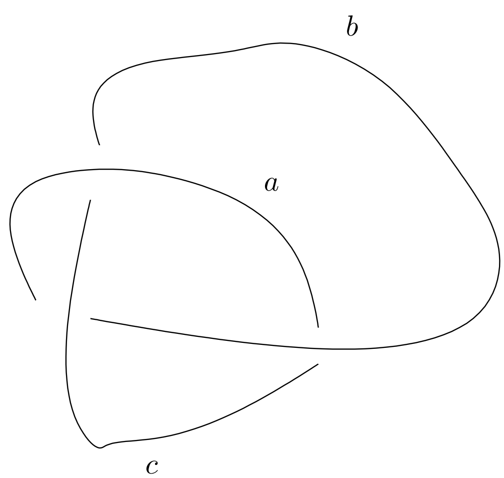
Let $n = 3, 4, \cdots$. A knot (or a link) is *n-colorable* if each strand can be labeled with a number $0, 1, 2, \cdots, n-1$ with the following rules:

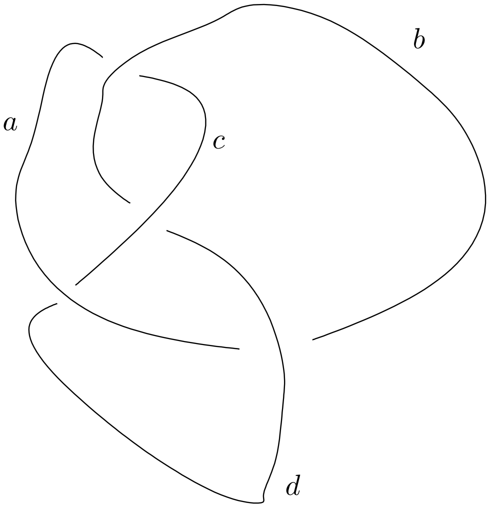
- At least two different numbers are used.
- At each crossing, if a is the label of the strand that is crossing over, and b, c are the labels of the other two strands, then

$$2a \equiv b + c \bmod n$$

9. Convince yourself that the case $n = 3$ amounts to tricolorability.

10. Imitate the proof that tricolorability is invariant under Reidemeister moves and show that n-colorability is also invariant under Reidermeister moves.





In the case of the trefoil, with labels as in the picture, the equations corresponding to the 3 crossings are $2a = b + c$, $2b = a + c$, $2c = a + b$. Notice that one equation is redundant since the sum reads $2a + 2b + 2c = 2a + 2b + 2c$ (one equations is always redundant). Solving for c in the second equation and substituting in the first, we get $3a = 3b$ (mod n of course). If n is not divisible by 3 this implies $a \equiv b$ and similarly $a \equiv c$, so that the trefoil is not n-colorable. If n is divisible by 3, say $n = 3k$, then $3a \equiv 3b$ always holds when a, b are divisible by k and we are down to just one equation, say $2a = b + c$, and this equation has solutions mod n with not all a, b, c equal, e.g. $a = k$, $b = 0$, $c = 2k$. We conclude that the trefoil is n-colorable precisely when $n \equiv 0 \bmod 3$.

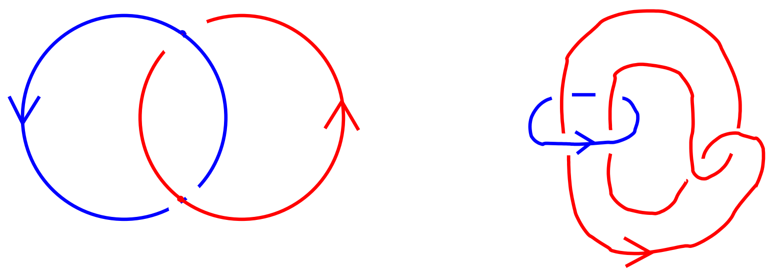
11. Show that the figure eight knot is n-colorable precisely when n is divisible by 5. Deduce that figure eight knot is a nontrivial knot.

12. Examine n-colorability for some of the knots in the list at the end of the handout. For example, show that 5_1 is n-colorable when $5|n$ and 5_2 is n-colorable when $7|n$. Is there a knot on the list which is not n-colorable for any $n \geq 3$, other than the unknot?

13. Prove that if a knot is n-colorable then its composition (connected sum) with any other knot is also n-colorable. Deduce that such knots do not have "inverses", i.e. the composition with any other knot is not an unknot.

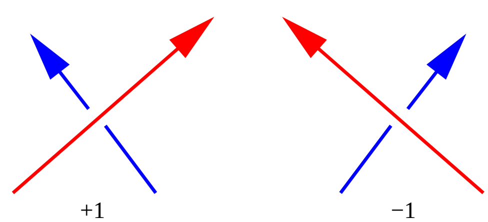
5 The linking number

Now let's think about 2-component links. We will color one component red and the other blue. We will also choose a sense of traversing each string (an *orientation* in the parlance of knot theory).



Hopf and Whitehead links

The idea is that we want to measure how many times one component "wraps around" the other. This is called the *linking number* and can be computed as follows. Look for those crossings where the red string is above the blue string. To each such crossing assign either a $+1$ or a -1 according to the right-hand rule ($+1$ if you can place the thumb of your right hand along the red string so that the other fingers point along the blue string; otherwise -1). The linking number is then equal to the sum of the numbers assigned to such crossings.



Right-handed rule

14. Find a sequence of Reidemeister moves showing that the above picture of the Whitehead link and 5^2_1 represent the same link.

15. Compute the linking number for the unlink of 2 components, for the Hopf link, and for the Whitehead link.

16. What happens to the linking number if we reverse the orientation of one of the components?

17. What happens to the linking number if we perform a Reidemeister move?

18. Conclude that the Whitehead and Hopf links are really different.

19. What happens to the linking number if we switch the colors? Hint: Look at it from behind.

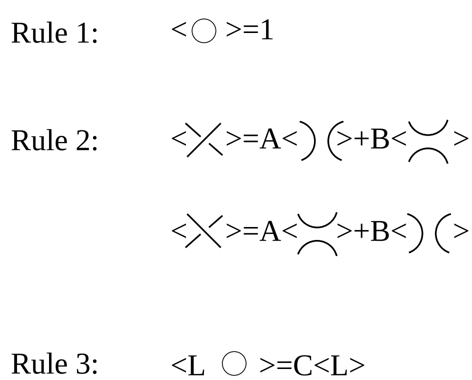
20. Examine the list of 2-component links at the end of the handout. Using the linking number and tricolorability, how many can you tell apart? For example 8^2_1 and 6^2_3 have different linking numbers.

21. Use linking numbers to show that 7^3_1 and 8^3_2 are different links. Find other pairs of 3-component links that you can tell apart.

6 The Jones' polynomial

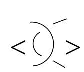
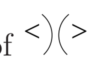
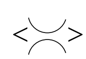
In the early 90's a mathematician at UC Berkeley named Vaughn Jones (a native of New Zealand) discovered a new way to tell knots apart. For this work he received the Fields Medal, the highest award in mathematics (equivalent to the Nobel Prize). He assigns a polynomial to every knot. If the polynomials are different, the knots are also different. It is possible for different knots to have the same Jones' polynomial, but it happens rarely. This polynomial is a remarkably good method of distinguishing knots. We will go through the construction of this polynomial that is due to Louis Kauffman.

The first step is to assign a *bracket* $< K >$ to every knot (or link) diagram K. This is going to be a polynomial, initially in variables A, B, C, and it will satisfy the following rules.



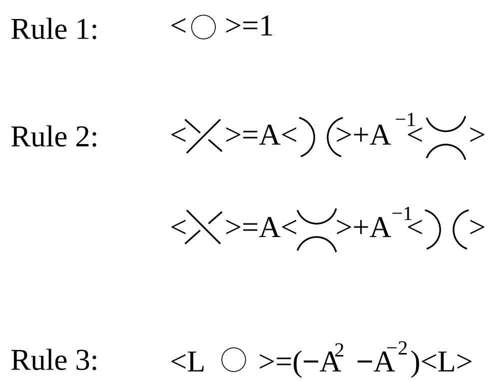
Bracket rules

Now we want to make sure that things don't change when we perform a Reidemeister move.

22. Using Rules 1-3 express  in terms of  and . Deduce: To ensure that the bracket does not change under the Reidemeister move of type II we need the following relations between A, B, C:

$$A^2 + ABC + B^2 = 0, \quad BA = 1$$

We will now put $B = A^{-1}$ and $C = -A^2 - A^{-2}$ and our Rules 1-3 become:



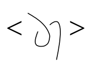
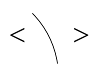
Bracket rules

23. Check that the bracket does not change after a Type III Reidemeister move.

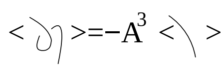
24. Compute the bracket of the unlink with 2 components and with 3 components.

25. Compute the bracket of the Hopf link. You should get $-A^4 - A^{-4}$.

There is still the Type I Reidemeister move to worry about.

26. Express  in terms of 

You should get



OK, so we have a problem! The bracket is not invariant under the Type I Reidemeister move. There are several possible remedies.

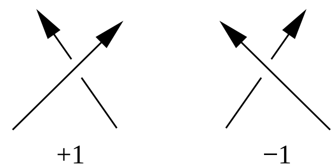
- We could find a number A so that $-A^3 = 1$. This would give us a numeric invariant of knots. For example $A = -1$ would work. There are more interesting choices for A; however, you would have to know about complex numbers.

- Another numeric invariant would be the *span* of the bracket polynomial, i.e. the difference between the highest and the lowest powers of A.

- The most interesting resolution of this problem involves the notion of the *writhe* of a knot projection.

7 Writhe

This is very similar to the concept of the linking number except that we are "color-blind". Orient all strings in the diagram and count with signs according to the right-hand rule.



The writhe

27. What is the writhe of the standard picture of the trefoil? What is the writhe of the unknot?

28. How does the writhe change if we reverse all orientations?

29. How does the writhe change under the Reidemeister moves?

30. Show that no matter how hard you try you will not be able to transform  into the circle (with no crossings) without using at least one Reidemeister move of Type I.

31. Show that

$$X(L) = (-A^3)^{-w(L)} < L >$$

is unchanged under all Reidemeister moves. Here L is a knot projection and $w(L)$ is the writhe of L.

32. Compute $X(L)$ for the trefoil.

33. Compute $X(L)$ for the figure 8 knot. Deduce that the figure 8 knot is really a knot!

34. What happens to $X(L)$ if we reverse all crossings?

Deduce that the trefoil is not the same as its mirror image. We say that trefoil is *chiral*. Recall that the figure 8 knot *is* the same as its mirror image. It is *achiral*.

The polynomial $X(L)$ is essentially the same as the original Jones' polynomial (replacing A by $t^{-\frac{1}{4}}$ would give the original version). You have seen that it is hard work to compute it, particularly if there are many crossings. But computers can handle this computation without much trouble. There is a program called *knotscape* (mathematicians' sense of humor) that performs this task.

8 Resources

- Colin C. Adams: *The knot book*, W.H. Freeman and Company, New York, 1999

  Most topics we discussed and many more are in this very accessible book. Pick up a copy and have fun!

- `http://knotplot.com`

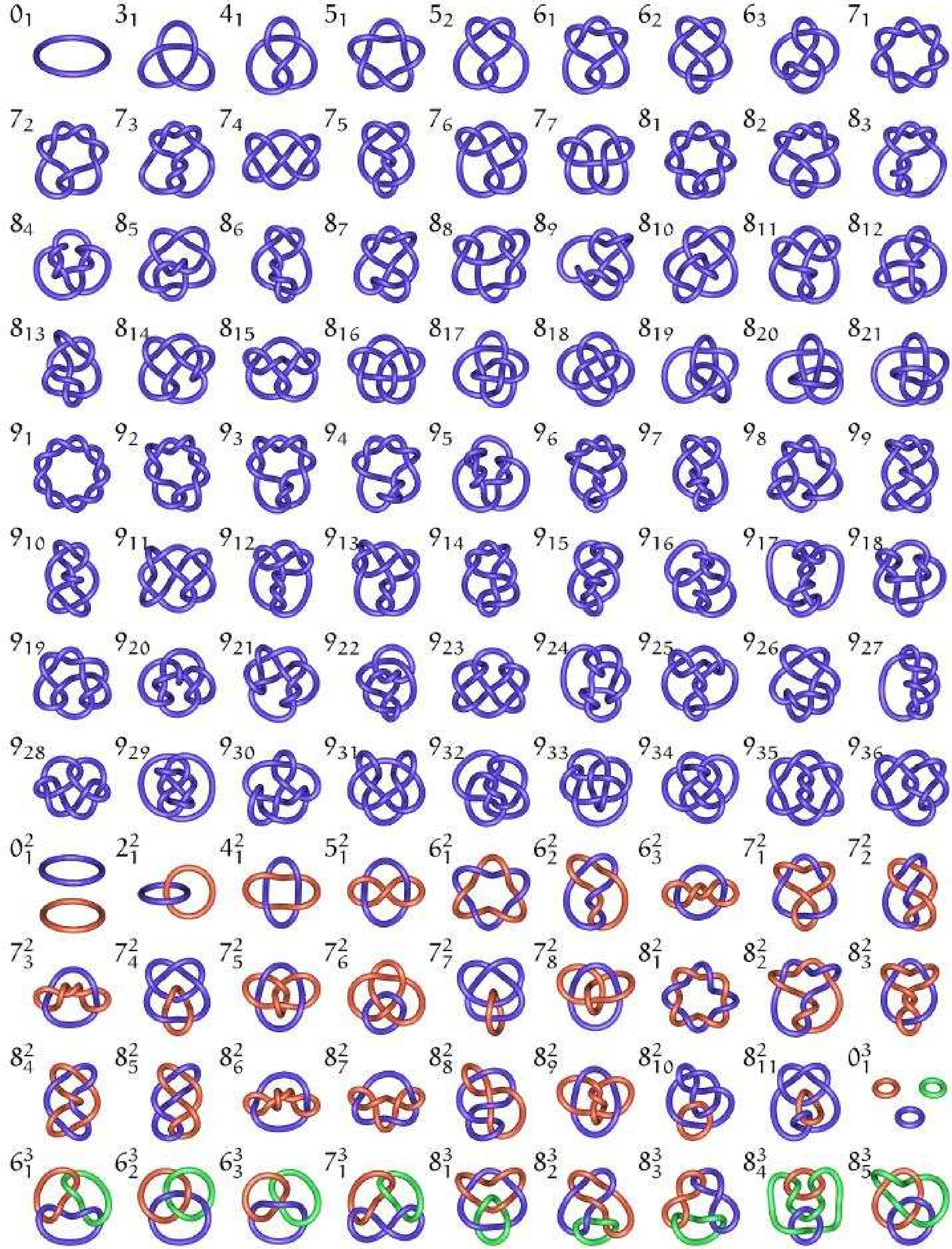
  Robert Scharein's excellent web site, packed with cool pictures, movies, and further links (no pun intended!). He is the author of *knotplot*, software that produces such pictures. You can download the program for free. The pictures on the next page are from this website.

- `http://www.math.utk.edu/~morwen/knotscape.html`

  Morwen Thistlethwaite's program that computes various knot polynomials. You draw a knot with the mouse and it computes the polynomials.

- `http://www.math.uic.edu/~kauffman/Tots/Knots.htm`

  Louis Kauffman's tutorial on knots and "bracketology". Try to read it. At least the first half should be accessible.


0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 7_1
7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2 8_3
8_4 8_5 8_6 8_7 8_8 8_9 8_{10} 8_{11} 8_{12}
8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20} 8_{21}
9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9
9_{10} 9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18}
9_{19} 9_{20} 9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27}
9_{28} 9_{29} 9_{30} 9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36}
0^2_1 2^2_1 4^2_1 5^2_1 6^2_1 6^2_2 6^2_3 7^2_1 7^2_2
7^2_3 7^2_4 7^2_5 7^2_6 7^2_7 7^2_8 8^2_1 8^2_2 8^2_3
8^2_4 8^2_5 8^2_6 8^2_7 8^2_8 8^2_9 8^2_{10} 8^2_{11} 0^3_1
6^3_1 6^3_2 6^3_3 7^3_1 8^3_1 8^3_2 8^3_3 8^3_4 8^3_5