

Math 6520 Final, Spring 2009

1. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$. Give an example of an irregular covering space $\tilde{X} \rightarrow X$. You should draw the picture of the 1-skeleton of \tilde{X} (make sure 2-cells lift).
2. Compute the fundamental group of the space obtained from the disjoint union of two 2-tori by identifying them along a pair of points.
3. Let S be an embedded circle in $\mathbb{R}P^2$ which is not null-homotopic (up to isotopy there is only one – the 1-skeleton of the standard cell structure) and let X be obtained by taking two copies of $\mathbb{R}P^2$ and identifying them along S . Compute the homology and cohomology groups of X with coefficients in \mathbb{Z} and \mathbb{Z}_2 (you can use any method you like, e.g. cell structure, Δ -structure, Mayer-Vietoris, long exact sequence of a pair, or Section 0 of Hatcher). For extra credit compute the cup product structure.
4. Let $f : \tilde{X} \rightarrow X$ be a covering map between connected nice spaces (e.g. manifolds or cell complexes). Suppose that f is null-homotopic. Show that \tilde{X} is contractible.
5. Let M be a closed connected 5-manifold and assume that $\pi_1(M) = \mathbb{Z}_3$ and $H_2(M; \mathbb{Z}) = 0$. Compute $H_i(M; \mathbb{Z})$ for all i . Hint: Use Poincaré duality (first argue M is orientable) and the universal coefficient theorem.
6. Let $f : \tilde{X} \rightarrow X$ be a 2-sheeted covering space between cell complexes. Show that there is a long exact sequence

$$\cdots \rightarrow H_i(\tilde{X}) \rightarrow H_i(X) \rightarrow H_{i-1}(X) \rightarrow H_{i-1}(\tilde{X}) \rightarrow \cdots$$

with all homology groups with coefficients in \mathbb{Z}_2 and use it to give another calculation of $H_i(\mathbb{R}P^n; \mathbb{Z}_2)$. Hint: Construct a short exact sequence $0 \rightarrow C_i(X; \mathbb{Z}_2) \rightarrow C_i(\tilde{X}; \mathbb{Z}_2) \rightarrow C_i(X; \mathbb{Z}_2) \rightarrow 0$ of cellular chain complexes. This is called the *transfer sequence* and the homomorphism $H_i(X) \rightarrow H_i(\tilde{X})$ above (that goes in the “wrong” direction) is the *transfer homomorphism*.

7. Let S be a genus 2 surface and $C \subset S$ a separating circle in S so that S/C is the wedge of two tori. Study the quotient map and your knowledge of the cup product structure on a torus to compute the cohomology ring of S .

8. Prove that there is no map $f : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2$ of negative degree. Hint: $f^* : H^4(\mathbb{C}P^2; \mathbb{Z}) \rightarrow H^4(\mathbb{C}P^2; \mathbb{Z})$ is multiplication by $\deg f$. Use the cup product structure.