Take Home Final

1. Let X be the space obtained from the 3-sphere S^3 by identifying antipodal points on the equator S^2. Compute $\pi_1(X)$.

2. Let $p : X \to Y$ be a covering map between path-connected and locally path-connected spaces. Suppose p is homotopic to a constant map. Prove that X is contractible.

3. Let $p : \tilde{X} \to X$ be a d-sheeted covering map for some $1 \leq d < \infty$.

   (a) Show that for every singular simplex $\sigma : \Delta^n \to X$ there are exactly d lifts $\tilde{\sigma}_i : \Delta^n \to \tilde{X}$.

   (b) Show that the homomorphism $\tau_n : C_n(X) \to C_n(\tilde{X})$ defined by
   $$\tau_n(\sigma) = \tilde{\sigma}_1 + \cdots + \tilde{\sigma}_d$$
   is a chain morphism $C_n(X) \to C_n(\tilde{X})$.

   (c) Show that the induced homomorphism
   $$\tau_* : H_n(X) \to H_n(\tilde{X})$$
   satisfies
   $$p_*\tau_*(x) = dx$$
   for every $x \in H_n(X)$. (τ_* is called *transfer*, a generic name for homomorphisms that go *the wrong way*).

   (d) Assuming $H_n(X), H_n(\tilde{X})$ are finitely generated, prove that
   $$rank(H_n(\tilde{X})) \geq rank(H_n(X))$$
   . Thus passing to a finite-sheeted covering space can only increase homology.

4. Let X be the space obtained from two copies of $\mathbb{R}P^2$ by gluing them along standard copies of $\mathbb{R}P^1$.

   (a) State Seifert-van Kampen's theorem and use it to compute $\pi_1(X)$.

   (b) Using any method you like (Δ-homology, cellular homology, Mayer-Vietoris,...) compute homology and cohomology groups of X with both $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients.

5. Prove that every map $S^4 \to S^2 \times S^2$ has degree 0, i.e. the induced homomorphism in H_4 is 0.

6. Prove that no closed orientable 3-manifold is homotopy equivalent to $S_g \vee S^3$, where S_g is the orientable surface of genus $g \geq 0$.