Cohomology

1. Let $p : \tilde{X} \to X$ be a double cover. Show that there is a long exact sequence (the Gysin sequence) of the form

$$\cdots \to H^k(X;\mathbb{Z}/2\mathbb{Z}) \to H^k(\tilde{X};\mathbb{Z}/2\mathbb{Z}) \to H^k(X;\mathbb{Z}/2\mathbb{Z}) \to H^{k+1}(X;\mathbb{Z}/2\mathbb{Z}) \to \cdots$$

   From this deduce that $H^i(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 0, 1, \cdots, n$ and is 0 otherwise.

   Hint: $0 \to C_n(X;\mathbb{Z}/2\mathbb{Z}) \to C_n(\tilde{X};\mathbb{Z}/2\mathbb{Z}) \to C_n(X;\mathbb{Z}/2\mathbb{Z}) \to 0$.

2. Let X be a space and suppose that $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, $H_1(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, $H_3(X;\mathbb{Z}) \cong \mathbb{Z}$ and $H_i(X;\mathbb{Z}) = 0$ for $i > 3$. Compute $H_i(X;\mathbb{Z}/2\mathbb{Z})$, $H^i(X;\mathbb{Z})$, $H^i(X;\mathbb{Z}/2\mathbb{Z})$ for all i.

3. Compute the ring structure on $H^*(X;\mathbb{Z}/n\mathbb{Z})$ where X is the Moore space $X = S^1 \cup_f e^2$ with the attaching map f of degree n. Hint: First compute cohomology groups $H^i(X;\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ for $i = 1, 2$ using the cellular chain complex. Then for the cup product use a Δ-complex structure on X and the definition of the cup product. Instead of doing a lot of linear algebra find a 1-cycle and a 1-cocycle whose Kronecker pairing over $\mathbb{Z}/n\mathbb{Z}$ is $1 \in \mathbb{Z}/n\mathbb{Z}$. This gives a generator in $H^1(X;\mathbb{Z}/n\mathbb{Z})$. Note that the case $n = 2$ recovers $H^*(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$.

4. Let X, Y be connected Δ-complexes and consider $Z = X \vee Y$, the wedge sum along vertices in X and Y. Show, by using functoriality properties and the definition of the cup product (just the easy part that if ϕ, ψ are cocycles that don't evaluate 0 on the faces of the same simplex then $\phi \cup \psi = 0$), that the cohomology ring $H^*(Z;\mathbb{Z})$ is obtained by taking the direct sum $H^*(X;\mathbb{Z}) \oplus H^*(Y;\mathbb{Z})$ of the two rings (with the product of elements in distinct coordinates 0) and then identifying the units $1_X = 1_Y$. Hint: Consider inclusions $X, Y \hookrightarrow Z$ and retractions $Z \to X, Y$.

5. Consider the map $S_2 \to T_1 \vee T_2$ from the genus 2 surface to the wedge of two tori obtained by collapsing a separating curve. Use your knowledge of the ring structure on the cohomology of the torus, Exercise #4, and cellular chain complexes to deduce the cohomology ring of S_2. Cf. Exercise #1 in Hatcher Section 3.2.

   In addition, do #6,7,10,11,18 from Hatcher Section 3.2.