Covering spaces

1. Let (X, Y) be a CW pair with both X, Y connected and $x_0 \in X$ a basepoint. Assume that inclusion induced homomorphism $\pi_1(Y, x_0) \to \pi_1(X, x_0) = G$ is injective and denote its image by $H < G$. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be the universal cover and let $p_H : (\tilde{X}_H, \tilde{x}_H) \to (X, x_0)$ be the covering corresponding to H.

   (a) Prove that each component of $p^{-1}(Y)$ is simply connected (and hence is a universal cover of Y).

   (b) Show that there is a natural bijection between the cosets gH of H in G and the components of $p^{-1}(Y)$ given by path lifting (which you should describe).

   (c) Assume that H is *malnormal* in G, i.e. $g \notin H$ implies $gHg^{-1} \cap H = \{1\}$. For example, $\langle a \rangle < \langle a, b \rangle$ is malnormal, and so is any subgroup of the fundamental group of an orientable surface defined by an embedded circle that doesn't bound a disk (this is called an *essential simple closed curve*). Show that one of the components of $p_H^{-1}(Y)$ is homeomorphic to Y and the others are simply-connected, and that the set of all components is in natural bijection with the set of double cosets HgH.

2. Do the following problems from Hatcher, Section 1.3: 4,8,9,12,14,18,20,21,30. On # 8 the hint should refer to Exercise 11 in Chapter 0. I am not going to assign problems regarding point set topology, but you may want to go over problems 5-7 for counterexamples to standard theorems when assumptions like local path connectedness are dropped.

3. This is optional, but if you are interested in geometry/topology you should read sections 1A and 1B in Hatcher. In particular, read the construction of the contractible Δ-complex EG on which a group G acts freely (in algebra, EG – or more precisely its chain complex – is called the *bar resolution* of G).