Seifert-van Kampen

1. Let X, Y be spaces and $f : X \to Y$ a map. The *mapping cone* of f is the space C_f obtained from the mapping cylinder $X \times I \sqcup Y/(x, 1) \sim f(x)$ by collapsing $X \times \{0\}$ to a point.

   (a) Now assume X, Y are path-connected. Express $\pi_1(C_f)$ as a pushout involving $\pi_1(X), \pi_1(Y)$ and $f_* : \pi_1(X) \to \pi_1(Y)$.

   (b) Let $X = Y = S^1 \vee S^1$ with the 1-cells labeled a, b and let $f : X \to Y$ be given by $a \mapsto aba^{-1}b^{-1}$ and $b \mapsto a^2b^3$. Compute $\pi_1(C_f)$ explicitly.

2. Let X be a space and $f : X \to X$ a map. The *mapping torus* of f is the space

$$T_f = X \times I/(x, 1) \sim (f(x), 0)$$

   Now suppose that X is a connected cell complex with $x_0 \in X$ the only 0-cell and $f : X \to X$ is a cellular map with $f(x_0) = x_0$. Then T_f is also a cell complex with cells $e \times \{0\}$ and $e \times (0, 1)$ for cells e of X (you don't have to prove this). Assume that $\pi_1(X, x_0) = \langle A \mid R \rangle$. Prove that

$$\pi_1(T_f) = \langle A, t \mid R, tat^{-1} = f_*(a), a \in A \rangle$$

   Hint: t corresponds to the loop $\{x_0\} \times I$. N.B.: The assumption that there is only one 0-cell is for convenience, the general case can be reduced to this.

3. Let A, B be nontrivial groups. Prove that $A * B$ has trivial center.

4. Let X be the space obtained from S^2 by identifying 3 points. Find an explicit cell structure and from it compute $\pi_1(X)$.

The following questions are harder (and more rewarding).

5. Problem 9 from Hatcher, p.53.

6. Problem 10 from Hatcher, p.53. Hint: Look at Problem 22 for inspiration.

7. Problem 14 from Hatcher, p.54.

8. Problem 15 from Hatcher, p.54.

9. Problem 16 from Hatcher, p.54. Hint: Show that the surface deformation retracts to a graph by representing it as an increasing union of compact subsurfaces $X_1 \subset X_2 \subset \cdots$ and showing that X_{n+1} deformation retracts to X_n union a graph.