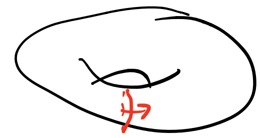
Poincaré Duality

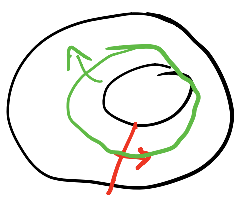


M^n closed oriented $\Rightarrow$ $H^i(M;\mathbb{Z}) \cong H_{n-i}(M;\mathbb{Z})$

Lefschetz Duality

M^n compact oriented $\Rightarrow$ $H^i(M,\partial M;\mathbb{Z}) \cong H_{n-i}(M;\mathbb{Z})$

$$H^i(M;\mathbb{Z}) \cong H_{n-i}(M,\partial M;\mathbb{Z})$$

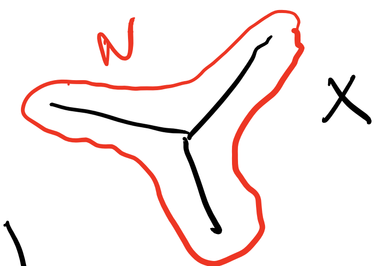


Application of LD

Alexander Duality: $X \subset S^n$ compact PL subset of S^n

Then $\tilde{H}_{i-1}(S^n - X) \cong \tilde{H}^{n-i}(X)$

Pf. X has regular nbhd N



$N \simeq \mathrm{Map}(\partial N \to X)$
$S^n - N \simeq S^n - X$

$$H_i(S^n, S^n - X) \simeq H_i\left(S^n, \overline{S^n - N}\right) \overset{exc}{\cong} H_i(N,\partial N)$$

$$\overset{LD}{\cong} H^{n-i}(N) = H^{n-i}(X)$$

LES of $(S^n, S^n - X)$: $H_i(S^n, S^n - X) \cong H_{i-1}(S^n - X)$

except $i = 1, n$. $\square$

Goal: Describe the PD isomorphism.

The Cap Product

$$\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

$k \geq \ell$.

$$\sigma \frown \varphi = \underbrace{\varphi\left(\sigma|_{[v_0,\dots,v_\ell]}\right)}_{\in R} \cdot \sigma|_{[v_\ell,\dots,v_k]}$$

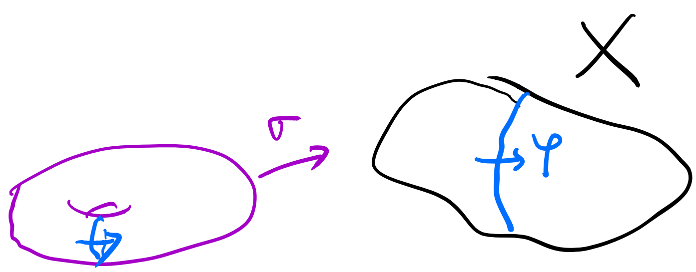
Prop. $\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$

So $\frown$ induces

$$H_k(X;R) \times H^\ell(X;R) \to H_{k-\ell}(X;R)$$

R-linear in each variable.

Geometrically



$\sigma \frown \varphi$ is represented by $\sigma|_{\sigma^*(\varphi)}$

Relative Versions

$$H_k(X,A) \times H^\ell(X) \longrightarrow H_{k-\ell}(X,A)$$
$$H_k(X,A) \times H^\ell(X,A) \longrightarrow H_{k-\ell}(X)$$

Naturality $\quad f: X \longrightarrow Y \qquad \alpha \in H_*(X)$
$$\varphi \in H^*(Y)$$

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

Also $\quad \alpha \in C_{k+\ell},\ \varphi \in C^k,\ \psi \in C^\ell$

$$\psi(\underbrace{\alpha \frown \varphi}_{\in C_k}) = (\underbrace{\varphi \smile \psi}_{\in C^{k+\ell}})(\alpha)$$

Thm (PD) M closed connected R-oriented n-mfld.

(1) There is a unique class $[M] \in H_n(M;R)$ that induces local orientations at every point

$$H_n(M;R) \longrightarrow H_n(M, M-x; R) \cong R$$
$$[M] \longmapsto \text{orient. class at } x.$$

[the fundamental class]

(2) $$H^i(M;R) \xrightarrow[\cong]{PD} H_{n-i}(M;R)$$

$$[\varphi] \longmapsto [M] \cap [\varphi] \quad \forall i.$$

<u>Cor.</u> (M connected) $H_n(M;R) \cong R$ generated by $[M]$

<u>Pf</u> $[1] \longmapsto [M] \cap [1] = [M]$

<u>Addendum</u> M connected, but either non-orientable, or not closed $\Rightarrow H_n(M;\mathbb{Z}) = 0$.

<u>Application</u> Compute the cohomology ring of $\mathbb{C}P^n$.

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\beta]/\beta^{n+1}=0$$

$\uparrow$ deg 2

Induction on n. By considering inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ have $\beta \in H^2(\mathbb{C}P^n)$ s.t. $1, \beta, \dots, \beta^{n-1}$ generate $H^0, H^2, \dots, H^{2n-2}$.

Need to see that β^n generates H^{2n}.

$$\psi(\alpha \cap \varphi) = (\varphi \cup \psi)(\alpha), \qquad \begin{aligned} \alpha &= [M] \\ \varphi &= \beta \\ \psi &= \beta^{n-1} \end{aligned}$$

$$\beta^{n-1}(\underbrace{[M] \cap \beta}_{PD(\beta)}) = (\beta \cup \beta^{n-1})[M]$$

$\leftarrow$ generates H_{2n-2}

$\therefore$ LHS $= \pm 1$

$\therefore$ β^n generates H^{2n} $\square$

More generally, intersection pairing

$$H^i(M) \times H^{n-i}(M) \xrightarrow{I} \mathbb{Z}$$

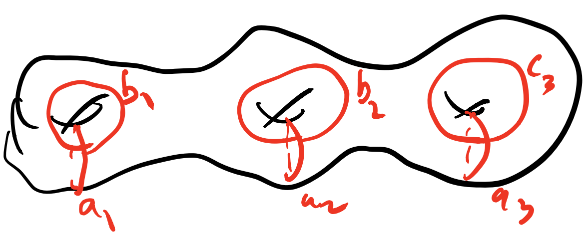
$$(\varphi, \psi) \longmapsto (\varphi \smile \psi)[M]$$

Really, this factors through

$$H^i/\text{torsion} \times H^{n-i}/\text{torsion} \longrightarrow \mathbb{Z}.$$

<u>Thm</u> This pairing is <u>unimodular</u>, i.e. if we choose a basis $\varphi_1, \dots, \varphi_k$ of $H^i/\text{torsion}$ and $\psi_1, \dots, \psi_k$ of $H^{n-i}/\text{torsion}$ then the matrix $\left(I(\varphi_i, \psi_j) \right)_{i,j=1}^{k}$ has $\det = \pm 1$.

<u>Ex.</u> $M = S_g$



$$\begin{array}{c} a_1 \; b_1 \;\; a_2 \; b_2 \;\; \cdots \\ \left(\begin{array}{cc|cc|cc} 0 & 1 & & 0 & & 0 \\ -1 & 0 & & & & \\ \hline & 0 & 0 & 1 & & 0 \\ & & -1 & 0 & & \\ \hline & 0 & & 0 & 0 & 1 \\ & & & & -1 & 0 \end{array} \right) \end{array}$$

Pf. $$(\varphi \cup \psi)[M] = \psi([M] \cap \varphi)$$
$$= \psi(PD(\varphi))$$

PD iso $\Rightarrow$ reduces to showing that Kronecker pairing

$$H^i/_{\text{tor}} \times H_i/_{\text{tor}} \to \mathbb{Z}$$

is unimodular.

$$\mathrm{Hom}(H_i;\mathbb{Z}) \times H_i/_{\text{tor}} \xrightarrow{\text{eval.}} \mathbb{Z}$$

which is unimodular by choosing dual bases.

Towards a proof: Need to generalize PD to apply to open manifolds.

Compactly supported cohomology

X loc. cpct space

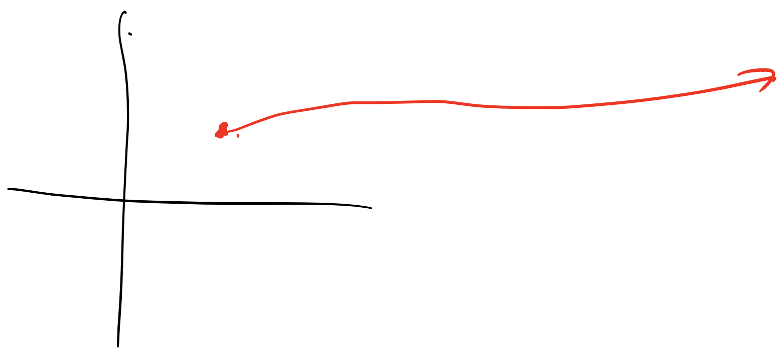
$$H^n_c(X) = \varinjlim_{K \subseteq X \text{ compact}} H^n(X, X-K)$$

Ex. $\mathbb{R}^n$

K ball in $\mathbb{R}^n$

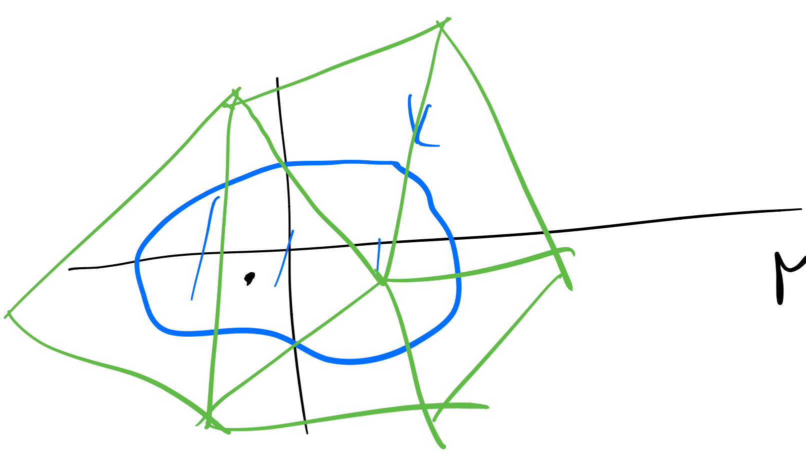
$$H^i(\mathbb{R}^n, \mathbb{R}^n - K) \overset{exc}{\cong} H^i(K, \partial K) = \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n \end{cases}$$

$$H^i_c(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n \end{cases}$$



Fund. class M^n oriented, not nec. compact.

For every compact $K \subseteq M$ there is a unique class $[M]_K \in H_n(M, M-K)$ that induces loc. orientation at every $x \in K$.



M

True for small K by def of orientation. For general K it's M-V.

By uniqueness, if $K \subset L$

$$H_n(M, M-L) \longrightarrow H_n(M, M-K)$$
$$[M]_L \longmapsto [M_K]$$

$PD : H^i_c(M) \longrightarrow H_{n-i}(M)$

$$(\varphi \in H^i(M, M-K)) \longmapsto [M]_K \cap \varphi \in H_{n-i}(M)$$

<u>Thm</u> PD is an isomorphism $\forall i$, $\forall M^n$ oriented.

<u>Pf.</u> M–V method. True for $M = \mathbb{R}^n$

Then follow the proof of de Rham's thm.