De Rham Theorem M smooth mfld then

$$H^i_{dR}(M) \cong H^i(M;\mathbb{R})$$

↑ singular cohomology.

Def. A cohomology theory on smooth mflds is a sequence of contravariant functors $\{h^n, n\in\mathbb{Z}\}$, $h^n : \{\text{smooth mflds, smooth maps}\} \to Ab$ with functorial homomorphism

$$\delta : h^n(U\cap V) \to h^{n+1}(M)$$

whenever $M = U\cup V$, U,V open s.t.

(1) $f,g : M \to N$ smoothly homotopic $\Rightarrow$ $f^* = g^* :$
$h^n(N) \to h^n(M)$ $\forall n$

(2) M-V: $M = U\cup V$

$$\cdots \to H^n(M) \xrightarrow{(i^*,-j^*)} H^n(U)\oplus H^n(V) \xrightarrow{+} H^n(U\cap V) \xrightarrow{\delta} H^{n+1}(M) \to \cdots$$

exact

(3) If $M = \coprod_\alpha U_\alpha$, $i_\alpha : U_\alpha \hookrightarrow M$ then $h^n(M) \xrightarrow{\cong} \prod_\alpha h^n(U)$

Thm Suppose k^*, h^* are two such cohomology theories, $\mu: h^*(M) \to k^*(M)$ natural transformation, $\mu: h^*(pt) \xrightarrow{\cong} k^*(pt)$. Then μ is an isomorphism $\forall M$.

There is a homological version, covariant functors, (3) is replaced by $\bigoplus h_n(U_\alpha) \xrightarrow{\cong} h_n(\bigsqcup U_\alpha)$.

A singular simplex $\sigma: \Delta^n \to M$ is smooth if σ can be extended to a smooth map of a nbhd of Δ^n in $\mathbb{R}^m \supset \Delta^n$.

$C_n^{smooth} \subset C_n$ subcomplex spanned by smooth n-simplices.

$\partial: C_n^{smooth} \longrightarrow C_{n-1}^{smooth}$

Thm Inclusion $C_n^{smooth} \hookrightarrow C_n$ induces an isomorphism in homology, as well as cohomology with any coefficients.

Pf. Suffices to prove this for homology with $\mathbb{Z}$ coeff. by the UCT.

Have $\mu : H_n^{smooth}(M) \to H_n(M)$ induced by inclusion.

μ is an iso for $M = pt.$ (all simplices are smooth).

Need to check that H_n^{smooth} satisfies the axioms.

Same pf as in the continuous case, just insert the word "smooth". ∎

Recall: $\Omega^i(M)$ = vect space of all smooth diff. i-forms on M.

$$d : \Omega^i(M) \to \Omega^{i+1}(M), \quad d^2 = 0.$$

$H^i_{dR}(M)$ = cohomology of the de Rham complex

$$0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0$$

<u>Key</u>:

$$\int : \Omega^i(M) \longrightarrow C^i_{smooth}(M;\mathbb{R})$$

$$\omega \longmapsto \left(\sigma \mapsto \int_\sigma \omega\right)$$

$\int$ is a cochain morphism by the Stokes theorem

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

So $\int$ induces $\qquad \int : H^i_{dR}(M) \to H^i_{smooth}(M;\mathbb{R})$

<u>Thm</u> This is an isomorphism $\forall M$.

$$H^i_{dR}(M) \xrightarrow{\cong} H^i_{smooth}(M;\mathbb{R}) \xleftarrow{\cong} H^i(M,\mathbb{R})$$

<u>Pf.</u> True for $M = pt$.
Need to check that H^*_{dR} satisfies the axioms.
This is 6510! $\square$

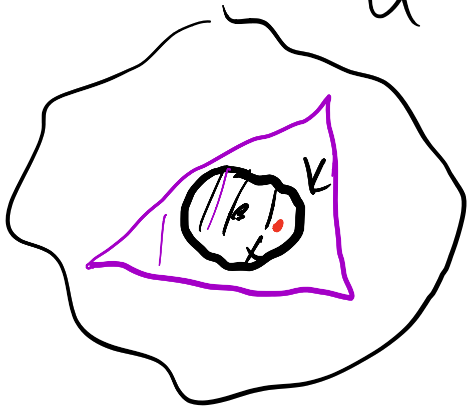
<u>Poincaré duality</u>

M closed (topological) n-mfld.

$R = \mathbb{Z}$ or $\mathbb{Z}_2$. If $R = \mathbb{Z}$ assume M is <u>oriented</u>.

In alg. top, local orientation (over R) at $x \in M$ is a choice of a generator of $H_n(M, M - x; R) \cong R$.
An orientation of M is a choice of a local orientation at every $x \in M$ which is locally constant.

$U \subseteq M$, $U \simeq \mathbb{R}^n$



$\exists$ compact nbhd $K \ni x$ in U and a sing simplex $\Delta^n \xrightarrow{\sigma} U$ with $\sigma(\partial \Delta^n) \cap K$ whose restriction to $(U, U-y)$ gives the loc. orientation at y, $y \in K$.

Last homework $\rightsquigarrow$ reconcile this definition of orientation with the one from smooth top.

Fundamental Class If M is oriented there is a unique class $[M] \in H_n(M;R)$ s.t. $H_n(M;R) \to H_n(M, M-x; R)$ takes $[M]$ to the local orientation.

This is the fundamental class.

Thm (Poincaré duality) M closed oriented over R, then $H^i(M;R) \cong H_{n-i}(M;R)$.

Some easy consequences

(0) M also connected $\Rightarrow$ $H_n(M;R) \cong R$

When $R = \mathbb{Z}$, it's generated by $[M]$.

① $b_i = \text{rank}(H_i(M))$ ith Betti number.

M orientable $\Rightarrow$ $b_i = b_{n-i}$

Pf. $b_{n-i} = \text{rk}(H_{n-i}(M)) \underset{UCT}{=} \text{rk}(H^{n-i}(M)) \underset{PD}{=} \text{rk}\, H_i(M) = b_i$.

Cor. n odd $\Rightarrow$ $\chi(M) = 0$.

$$\begin{array}{c}\chi(M) \\ \| \\ b_0 - b_1 + b_2 - \cdots \qquad + b_{n-1} - b_n\end{array}$$

Also true when M is not orientable.

Pf1: Use $\mathbb{Z}_2$-coeff.

Pf2: Use the orientation double cover

$$\begin{array}{ccc} \tilde{M} & \qquad & S^2 \\ \downarrow & & \downarrow \\ M & & \mathbb{R}P^2 \end{array}$$

$0 = \chi(\tilde{M}) = 2\chi(M) \Rightarrow \chi(M) = 0$.

② Ex. M^5 closed connected mfld.

Spse $H_1(M) = \mathbb{Z}_3$, $H_2(M) = \mathbb{Z}$.

Compute $H_i(M)$, $H^i(M)$ $\forall i$.

	H_*	H^*
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}_3$	0
2	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_3$
3	$\mathbb{Z}\oplus\mathbb{Z}_3$	$\mathbb{Z}$
4	0	$\mathbb{Z}_3$
5	$\mathbb{Z}$	$\mathbb{Z}$

UCT

PD

PD

M is orientable because nonorientable manifolds have epimorphism $\pi_1 M \to \mathbb{Z}_2$ (orientation double cover)

This factors through $H_1 M = \mathbb{Z}_3$, so M must be orientable.

(3) M^3 closed connected nonorientable. Then $b_1 > 0$.

Fact If M^n is closed connected nonorientable then $H_n(M) = 0$.

Pf of (3)

$$0 = \underset{\underset{1}{\parallel}}{b_0} - b_1 + b_2 - \underset{\underset{0}{\parallel \text{ Fact}}}{b_3}$$

$$b_1 = b_0 + b_2 - b_3 = 1 + b_2 \geq 1.$$

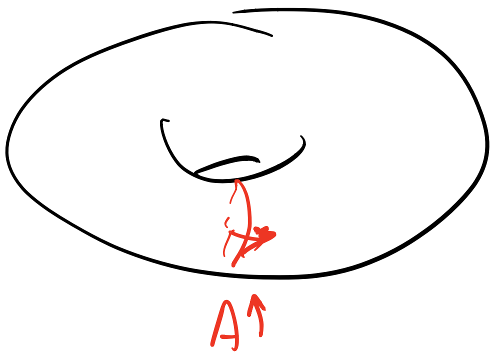
Relative version Lefschetz duality

M compact oriented n-mfld with ∂.

Then $H^i(M, \partial M) \cong H_{n-i}(M)$

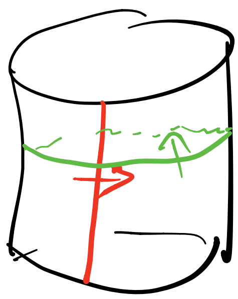
$$H^i(M) \cong H_{n-i}(M, \partial M).$$

<u>Geometrically</u>: An element of $H^i(M)$ is represented (usually) by a codim i submanifold of M



$[A] \longmapsto$ image of the fund. class of A under $H_{n-i}(A) \to H_{n-i}(M)$

$[A] \in H^i(M)$



M

$$H_1(M, \partial M) \leftarrow H^1(M)$$

red

$$H_1(M) \leftarrow H^1(M, \partial M)$$

green