The Mayer-Vietoris method

Def. A *cohomology theory* is a sequence of contravariant functors $h^n : \{\text{CW pairs}\} \to \{\text{abelian groups}\}$, $n \in \mathbb{Z}$, and functorial homomorphisms

$$\delta : \underset{(A,\phi)}{\overset{\shortparallel}{h^n(A)}} \longrightarrow h^{n+1}(X,A)$$

s.t.

(1) $f \simeq g : (X,A) \to (Y,B) \Rightarrow f^* = g^* : h^n(Y,B) \to h^n(X,A)$

(2) (excision) If $f : (X,A) \to (Y,B)$ induces a homeo $X/A \to Y/B$ then $f^* : h^n(Y,B) \xrightarrow{\cong} h^n(X,A)$

(3) LES

$$\cdots \longrightarrow h^n(X,A) \longrightarrow h^n(X) \longrightarrow h^n(A) \xrightarrow{\delta} h^{n+1}(X,A) \to \cdots$$

induced by inclusion.

is exact.

(4) $X = \coprod_\alpha X_\alpha$, $i_\alpha : X_\alpha \hookrightarrow X$ inclusion.

Then $\prod_\alpha i_\alpha^* : h^n(X) \xrightarrow{\cong} \prod_\alpha h^n(X_\alpha)$

<u>Thm</u> Let h^n, k^n be two cohomology theories and $\mu : h^*(X,A) \to k^*(X,A)$ a natural transformation. If μ is an isomorphism for $(X,A) = (pt, \phi)$, then it is an isomorphism $\forall (X,A)$.

<u>Pf</u> If $\dim X = 0$, follows from (4).

If $\dim X < \infty$, we argue by induction on $n = \dim X$.

From LES + 5-lemma, it suffices to prove it for (X, X^{n-1}).

Let $\Phi : \coprod (D^n_\alpha, \partial D^n_\alpha) \longrightarrow (X, X^{n-1})$ be the characteristic maps of the n-cells of X.

By excision Φ^* is an iso in both h^*, k^*.

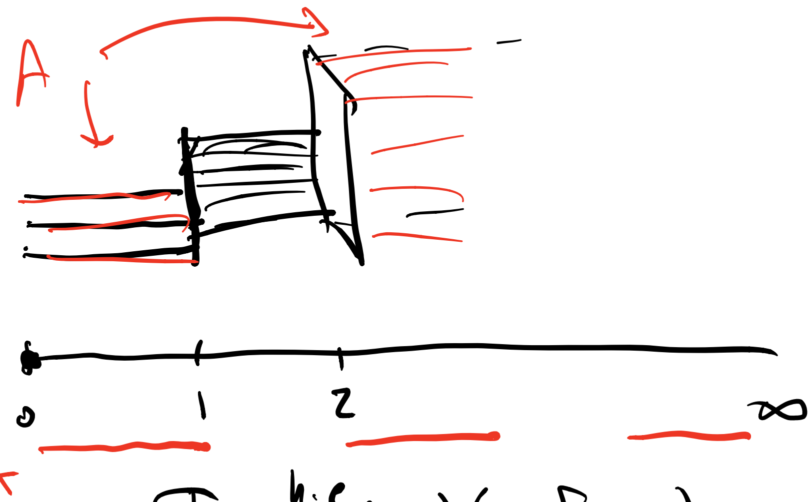
By (4) we are reduced to proving the claim for the pair $(D^n, \partial D^n)$.

But this follows from LES, 5-lemma, induction and the homotopy axiom $D^n \simeq *$.

→ so also true for pairs (X,A) if $\dim X < \infty$

If $\dim X = \infty$, use the "telescoping argument":

$$T \subset X \times [0,\infty), \quad T = \bigcup_{i=0}^{\infty} X^i \times [i,\infty)$$



$$T \simeq X, \quad T \xhookrightarrow{h.e.} X \times [0,\infty)$$

Let $A = \bigcup_{i \text{ even}} X^i \times [i,i+1]$

The statement holds for A, since $A = \coprod$ (finite dim'l CW cxs)

To prove the statement for X, it suffices to prove it for T, and by LES, for (T,A).

But by excision $(T,A) \underset{exc}{\simeq} \coprod_{i \text{ odd}} \left(X^i \times [i,i+1], X^i \times \{i,i+1\}\right)$

But RHS is a disjoint union of finite dim'l pairs. $\square$

Künneth formula Y cw ex, $H^n(Y;R)$ f.g. free R-module $\forall n$

Then $\displaystyle\bigoplus_{i+j=n} H^i(X,A;R) \otimes_R H^j(Y;R) \xrightarrow[\cong]{\times} H^n(X\times Y, A\times Y;R)$

((X,A) cw pair)

Pf. $h^n(X,A) := \displaystyle\bigoplus_{i+j=n} H^i(X,A;R) \otimes_R H^j(Y,R)$

$k^n(X,A) := H^n(X\times Y, A\times Y;R)$

$\mu : h^n(X,A) \to k^n(X,A)$ cross product.

To check: h^*, k^* are cohomology theories

- μ is clearly an iso for $(X,A)=(pt,\phi)$

for k^* this is easy.

for h^* need to use the assumption on Y.

LES:

$\to H^i(X,A) \to H^i(X) \to H^i(A) \to H^{i+1}(X,A) \to \dots$

$\otimes_R H^j(Y)$:

$\to H^i(X,A) \otimes_R H^j(Y) \to H^i(X) \otimes_R H^j(Y) \to \dots$

still exact by the assumption that $H^j(Y)$ is a free module

Take a direct sum for $i+j=n$

Disj. union $\left(\coprod_\alpha M_\alpha\right) \otimes N \cong \coprod_\alpha (M_\alpha \otimes N)$

$N = H^n(Y)$

$N = \mathbb{R}^2$ $\quad LHS = \left(\coprod_\alpha M_\alpha\right) \times \left(\coprod M_\alpha\right)$

$\quad\quad\quad\quad\; \shortparallel$

$\quad\quad\quad\; RHS = \coprod_\alpha (M_\alpha \times M_\alpha)$

$\square$

De Rham Theorem $\quad M$ smooth mfld then

$$H^i_{dR}(M) \cong H^i(M; \mathbb{R})$$

$\uparrow$ singular cohomology.

Def. A cohomology theory on smooth mflds is a sequence of contravariant functors $\{h^n, n \in \mathbb{Z}\}$, $h^n : \{\text{smooth mflds, smooth maps}\} \to Ab$ with functorial homomorphism

$$h^n(M) \leftarrow h^{n-1}(U \cap V) : \delta$$

whenever $M = U \cup V$, U, V open s.t.

(1) $f, g : M \to N$ smoothly homotopic $\Rightarrow f^* = g^* :$ $h^n(N) \to h^n(M)$ $\forall n$

(2) M–V: $M = U \cup V$

$$\cdots \to H^n(M) \xrightarrow[(i^*,-j^*)]{} H^n(U) \oplus H^n(V) \xrightarrow[+]{} H^n(U \cap V) \xrightarrow{\delta} H^{n+1}(M) \to \cdots$$

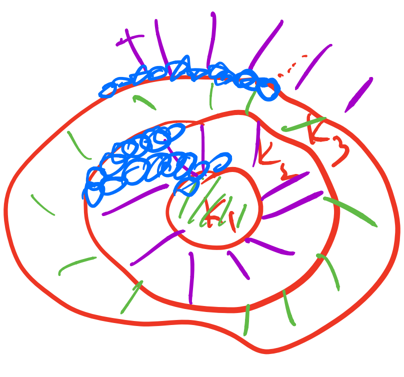
exact

(3) If $M = \coprod_\alpha U_\alpha$, $i_\alpha : U_\alpha \hookrightarrow M$ then $h^n(M) \xrightarrow{\cong} \prod_\alpha h^n(U_\alpha)$

Thm Suppose k^*, h^* are two such cohomology theories, $\mu : h^*(M) \to k^*(M)$ natural transformation, $\mu : h^*(pt) \xrightarrow[\cong]{} k^*(pt)$. Then μ is an isomorphism $\forall M$.

Pf True for M = finite union of open convex sets in $\mathbb{R}^n$ by induction & M–V.

Let $U \subseteq \mathbb{R}^n$ be open. Let $K_1 \subseteq K_2 \subseteq \cdots$ be an exhaustion of U by compact sets K_i, $K_i \subseteq \operatorname{int} K_{i+1}$



Let $A_i := \overline{K_{i+1} - K_i}$

Then A_i, A_j disjoint $|i-j|>1$

Cover each A_i by finitely many open convex sets whose union is U_i and so that $U_i \cap U_j = \emptyset$, $|i-j|>1$

Then this is true for each U_i, also for $U_i \cap U_{i+1}$.

Let $U_{even} = \coprod_{i \text{ even}} U_i$, $U_{odd} = \coprod_{i \text{ odd}} U_i$.

True for U_{even}, U_{odd}, $U_{even} \cap U_{odd}$. So true for U by M-V.

In general: M arbitrary smooth. Repeat the above argument replacing "convex" by a "chart".

$\square$