$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha]/_{\alpha^{n+1}=0}$, $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/_{\alpha^{n+1}=0}$

(α deg 2 ; α deg 1)

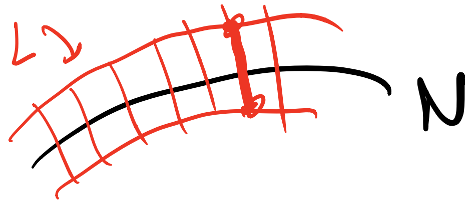
Recall M closed smooth n-mfd

oriented, $R=\mathbb{Z}$; otherwise $R=\mathbb{Z}_2$.

$N \subset M$ closed smooth submanifold of codim k.

$R=\mathbb{Z} \to N$ oriented normal bundle.

$\tau \in H^k(L,\partial L)$ Thom class, $L=$ tubular nbhd on N



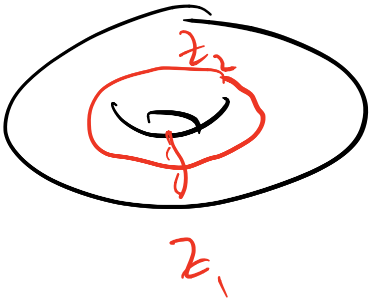
$$H^k(L,\partial L) \longrightarrow H^k(B^k,\partial B^k)=\mathbb{Z}$$
$$\uparrow \text{fiber}$$
$$\tau \longmapsto \text{generator}$$

$$\begin{array}{ccccc} H^k(L,\partial L) & \xleftarrow[\cong]{exc} & H^k(M,\overline{M-L}) & \longrightarrow & H^k(M) \\ \Psi & & & & \Psi \\ \tau & & \longmapsto & & PD(N) \end{array}$$

Poincaré dual of N.

Thm If $Z_1, Z_2 \subset M^n$ are two submanifolds as above of codim n_1, n_2 s.t. $Z_1 \cap Z_2 = 1$ pt, and intersection is transverse. Then $PD(Z_1) \cup PD(Z_2) \in H^n(M)$ $(n_1+n_2=n)$ generates $H^n(M) \cong \mathbb{Z}$ (or $\mathbb{Z}_2$)

Ex:



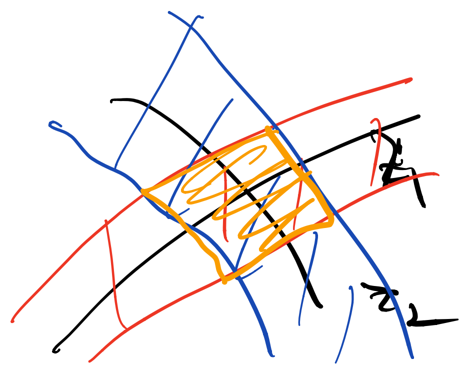
$\mathbb{C}P^i \pitchfork \mathbb{C}P^{n-i} \subset \mathbb{C}P^n$

one point

$\mathbb{R}P^i \pitchfork \mathbb{R}P^{n-i} \subset \mathbb{R}P^n$

Pf. N_i = reg. nbhd of z_i

$$\begin{tikzcd}
H^{n_1}(I^{n_1},\partial I^{n_1}) & \otimes & H^{n_2}(I^{n_2},\partial I^{n_2}) \arrow[r, "\times", "\cong"'] & H^{n}(I^{n_1}\times I^{n_2},\partial) \\
\tau_1 \in H^{n_1}(N_1,\partial N_1) \arrow[u, "\text{restr. to fiber}"', "\cong\ \text{Thom isomorphism thm.}"] & \otimes & H^{n_2}(N_2,\partial N_2) \ni \tau_2 \arrow[u, "\text{restr. to fiber}"', "\cong"] & \\
H^{n_1}(M,\overline{M-N_1}) \arrow[u, "\text{exc}"', "\cong"] \arrow[d] & \otimes & H^{n_2}(M,\overline{M-N_2}) \arrow[u, "\text{exc}"', "\cong"] \arrow[d] \arrow[r] & H^{n}(M,\overline{M-I^{n_1}\times I^{n_2}}) \arrow[uu, "\cong", "\text{exc}"'] \arrow[d, "\cong\ \text{general fact, true for } \mathbb{C}P^n,\ \mathbb{R}P^n \text{ by cellular cohomology}"] \\
H^{n_1}(M) & \otimes & H^{n_2}(M) \arrow[r, "\cup"] & H^{n}(M) \\
PD(N_1) \arrow[u, phantom, "\in"] & & PD(N_2) \arrow[u, phantom, "\in"] &
\end{tikzcd}$$



Application to $\mathbb{C}P^n$

$$H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/_{\alpha^{n+1}=0} = \langle \underset{\deg 0}{1}, \underset{\deg 2}{\alpha}, \underset{\deg 4}{\alpha^2}, \ldots, \underset{\deg 2n}{\alpha^n} \rangle$$

Pf by induction on n

$n=1$, $\quad H^*(S^2, \mathbb{Z}) = \langle 1, \underset{\deg 2}{\underset{\uparrow}{\alpha}} \rangle = \mathbb{Z}[\alpha]/_{\alpha^2=0}$

Consider inclusion $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$

By cellular cohomology $H^i(\mathbb{C}P^n) \xleftarrow{\cong} H^i(\mathbb{C}P^{n+1})$

$i = 0, 1, \ldots 2n.$

Denote by $\alpha \in H^2(\mathbb{C}P^{n+1})$ a generator.

Then by induction $1, \alpha, \ldots, \alpha^n$ generate

$H^0(\mathbb{C}P^{n+1}), H^2(\mathbb{C}P^{n+1}), \ldots, H^{2n}(\mathbb{C}P^{n+1}).$

We need to show that $\underset{\alpha \cup \alpha^n}{\overset{\parallel}{\alpha^{n+1}}}$ generates $H^{2n+2}(\mathbb{C}P^{n+1}).$

By the theorem, there is a class in H^2 and a class in H^{2n} whose cup product generates H^{2n+2}.

It follows that these classes are $\pm\alpha$ and $\pm\alpha^n$, So $\alpha \cup \alpha^n$ generates H^{2n+2}. $\square$.

- α is really $PD(\mathbb{C}P^{n-1})$

$\alpha^i = PD(\mathbb{C}P^{n-i})$

Pf for $\mathbb{R}P^n$ is similar.

Application:

<u>Borsuk-Ulam Theorem</u> There is no map $\mathbb{R}P^n \longrightarrow \mathbb{R}P^{n-1}$ which is nontrivial in π_1, $n \geq 2$.

<u>Pf.</u> $n = 2$

$$\begin{array}{ccc} \pi_1(\mathbb{R}P^2) & \longrightarrow & \pi_1(\mathbb{R}P^1) \\ \| & & \| \\ \mathbb{Z}_2 & \longrightarrow & \mathbb{Z} \end{array} \quad \checkmark$$

<u>$n > 2$</u> Then $f : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$ is an isomorphism in π_1, hence in $H_1(\ ;\mathbb{Z})$, hence in

$$H^1(\ ;\mathbb{Z}_2) = \mathrm{Hom}(H_1, \mathbb{Z}_2).$$

Let $\alpha_n \in H^1(\mathbb{R}P^n;\mathbb{Z}_2)$, $\alpha_{n-1} \in H^1(\mathbb{R}P^{n-1};\mathbb{Z}_2)$ be the generators,

Then $f^*(\alpha_{n-1}) = \alpha_n$

$$f^*(\underset{0}{\underset{\|}{\alpha_{n-1}^n}}) = \underset{\neq 0}{\alpha_n^n} \quad \times.$$

Another Version There is no $\mathbb{Z}_2$-equivariant map

$S^n \longrightarrow S^{n-1}$, wrt antipodal action.

Yet another version For every map $f: S^n \to \mathbb{R}^n$

$\exists\, x \in S^n$ s.t. $f(x) = f(-x)$.

Pf: Otherwise $h: S^n \to S^{n-1}$

$$x \longmapsto \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$$

violates the previous version.

The Mayer-Vietoris method

Def. A *cohomology theory* is a sequence of contravariant functors $h^n : \{\text{CW pairs}\} \to \{\text{abelian groups}\}$

$n \in \mathbb{Z}$, and functorial homomorphisms

$$\delta : \underset{\substack{\shortparallel \\ (A,\phi)}}{h^n(A)} \longrightarrow h^{n+1}(X, A)$$

s.t.

(1) $f \simeq g : (X, A) \to (Y, B) \Rightarrow f^* = g^* : h^n(Y,B) \to h^n(X,A)$

(2) (excision) If $f: (X,A) \to (Y,B)$ induces a

homeo $X/A \to Y/B$ then $f^*: h^n(Y,B) \xrightarrow{\cong} h^n(X,A)$

(3) LES

$$\cdots \to h^n(X,A) \to h^n(X) \to h^n(A) \xrightarrow{\delta} h^{n+1}(X,A) \to \cdots$$

induced by inclusion.

is exact.

(4) $X = \coprod_\alpha X_\alpha$, $i_\alpha : X_\alpha \hookrightarrow X$ inclusion.

Then $\prod_\alpha i_\alpha^* : h^n(X) \xrightarrow{\cong} \prod_\alpha h^n(X_\alpha)$

<u>Thm</u> Let h^n, k^n be two cohomology theories and $\mu : h^*(X,A) \to k^*(X,A)$ a natural transformation. If μ is an isomorphism for $(X,A) = (pt, \phi)$, then it is an isomorphism $\forall (X,A)$.