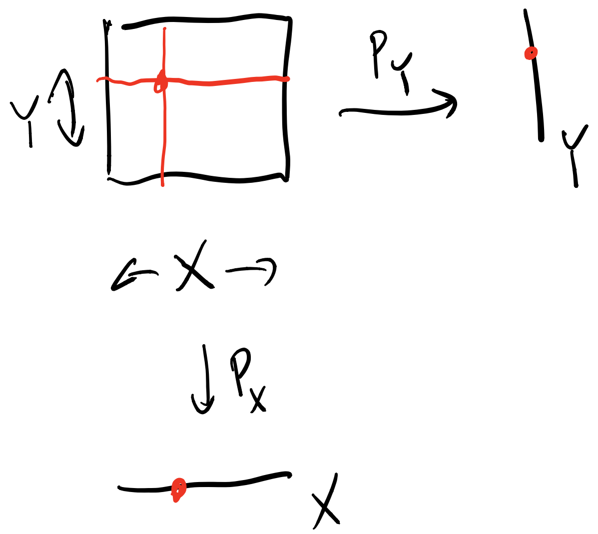
Cross Product and Computations

$$H^k(X;R)\otimes H^\ell(Y;R) \xrightarrow{\times} H^{k+\ell}(X\times Y;R)$$



$$x \times y = p_X^*(x)\cup p_Y^*(y)$$

Prop $(a\times b)\cup(c\times d) = (-1)^{\deg b\cdot\deg c}(a\cup c)\times(b\cup d)$

[Set theoretically: $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$]

Pf. $(a\times b)\cup(c\times d) = \big(p_X^*(a)\cup p_Y^*(b)\big)\cup\big(p_X^*(c)\cup p_Y^*(d)\big)$

$$= (-1)^{\deg b\cdot\deg c}\, p_X^*(a\cup c)\cup p_Y^*(b\cup d) = (-1)^{\deg b\cdot\deg c}(a\cup c)\times(b\cup d)$$

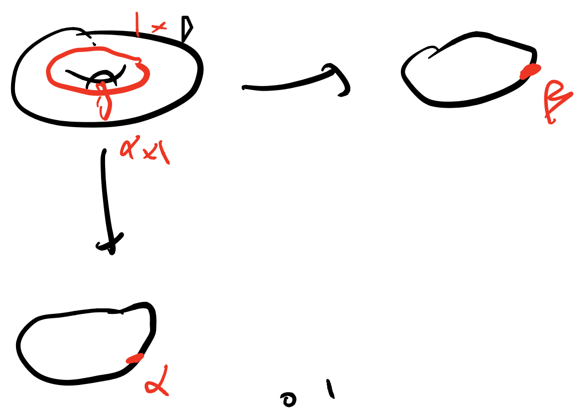
Künneth Formula Suppose $H^i(Y;R)$ is a free R-module of finite rank $\forall i$. Then $\forall X$

$$\bigoplus_{i+j=n}\left[H^i(X;R)\otimes H^j(Y;R)\right]\xrightarrow{\cong} H^n(X\times Y;R)$$

where the homomorphism on each summand is the cross product.

Pf postponed – Mayer-Vietoris argument.

Ex $T^2 = S^1\times S^1$



$H^* = \langle \overset{0}{1}, \overset{1}{\beta}\rangle$

$H^* = \langle \overset{0}{1}, \overset{1}{\alpha}\rangle$

Künneth $\Rightarrow$ $H^*(T^2)$ is freely generated by $1\times 1,\ 1\times\beta,\ \alpha\times 1,\ \alpha\times\beta$

Prop $\Rightarrow$ $(\alpha\times 1)\cup(1\times\beta) = \alpha\times\beta$

Thm $H^*(T^n) = \Lambda(\underset{\deg 1}{\alpha_1}, \underset{\deg 1}{\alpha_2}, \ldots, \underset{\deg 1}{\alpha_n})$ "exterior algebra"

$\alpha_i^2 = 0$

e.g. $H^2(T^3) = \langle \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_2\alpha_3\rangle \cong \mathbb{Z}^3$

$\alpha_i\alpha_j = -\alpha_j\alpha_i$

If α has odd degree then $\alpha \cup \alpha = -\alpha \cup \alpha$, so α^2 has order 1 or 2

Pf. Induction on n. Define $\alpha_i = p_i^*(\gamma)$,

$p_i : T^n \to S^1$ i-th coord.

$\gamma \in H^1(S^1)$ generator.

Use the Künneth formula.

$$T^3 = T^2 \times S^1$$

$$H^2(T^3) \cong \Big(\underbrace{H^2(T^2)}_{\alpha_1\alpha_2} \otimes \underbrace{H^0(S^1)}_{1}\Big) \oplus \Big(\underbrace{H^1(T^2)}_{\alpha_1,\alpha_2} \otimes \underbrace{H^1(S^1)}_{\alpha_3}\Big)$$

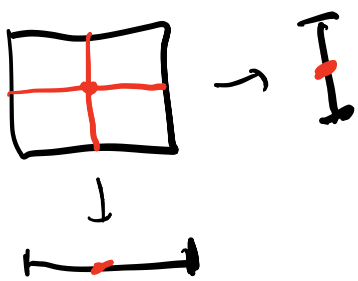
$\square$

Cor. $H^k(T^n) \cong \mathbb{Z}^{\binom{n}{k}}$

Next goal $\mathbb{C}P^n$, $\mathbb{R}P^n$ over $\mathbb{Z}_2$.

Key Fact $H^i(I^i, \partial I^i) \otimes H^j(I^j, \partial I^j) \overset{\times}{\cong} H^{i+j}(I^{i+j}, \partial I^{i+j})$

(relative cross product)



$i = j = 1$

One proof: Relative version of Künneth

Another proof: Compare with $T^i \times T^j \cong T^{i+j}$

$$(I^i, \partial I^i) \longrightarrow (T^i, T^{(i-1)})$$

$$\begin{array}{ccccc}
H^i(I^i, \partial I^i) \otimes H^j(I^j, \partial I^j) & \xrightarrow{\times} & H^{i+j}(I^{i+j}, \partial I^{i+j}) \\
\uparrow \qquad\qquad \uparrow & & \uparrow \\
H^i(T^i, T^{(i-1)}) \otimes H^j(T^j, T^{(j-1)}) & \xrightarrow{\times} & H^{i+j}(T^{i+j}, T^{(i+j-1)}) \\
\uparrow \qquad\qquad \uparrow & & \uparrow \\
H^i(T^i) \otimes H^j(T^j) & \xrightarrow[\cong]{\times} & H^{i+j}(T^{i+j})
\end{array}$$

All vertical maps are isos by cellular cohomology

Thom Isomorphism Thm

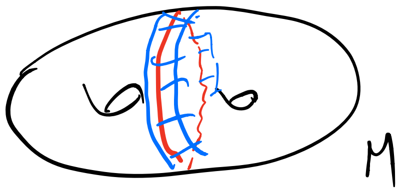
$(B^n, \partial B^n) \hookrightarrow (E, \partial E) \longrightarrow B$ bundle.

$R = \mathbb{Z}$ then assume the bundle is oriented, or $R = \mathbb{Z}_2$.

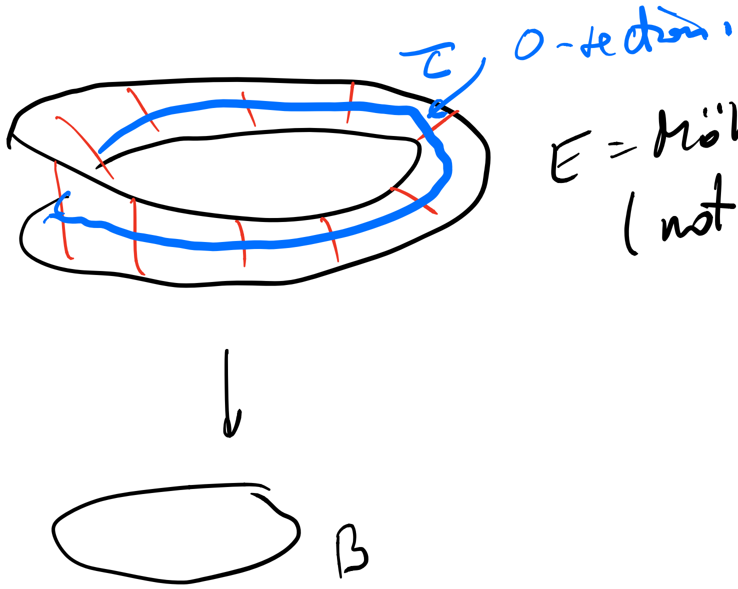
Thm (i) $\exists$ unique class $\tau \in H^n(E, \partial E, R)$, called the Thom class, s.t. the restriction to any fiber is the generator given by the orientation.

(ii) $H^k(B;R) \xrightarrow{\cong} H^{n+k}(E,\partial E;R)$

$x \longmapsto p^*(x)\cup\tau.$



E = tubular nbhd of a smooth submanifold $B \subset M$



E = Möb strip.
(not orientable, so $R = \mathbb{Z}_2$)

On the proof

- If the bundle is trivial, follows from Künneth.
- In general, proof by the "Mayer-Vietoris method".
- In applications we do in class, need to know this for normal bundle of $\mathbb{C}P^i \subset \mathbb{C}P^n$ (or $\mathbb{R}P^i \subset \mathbb{R}P^n$), and this we can prove using cellular cohomology.

Pf for $\mathbb{C}P^i \subset \mathbb{C}P^n$

N = tub-nbhd of $\mathbb{C}P^i$.

$\mathbb{C}P^n - \mathbb{C}P^i$ deformation retracts to $\mathbb{C}P^{n-i-1} \subset \mathbb{C}P^n$

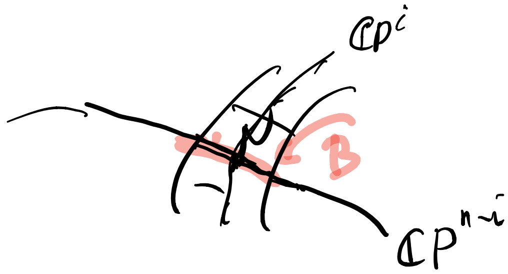
$$\mathbb{C}P^i = \{[X_0 : \cdots : X_i : 0 : \cdots : 0]\}$$

$$\mathbb{C}P^n - \mathbb{C}P^i = \{[X_0 : \cdots : X_n] \mid \text{not all } X_{i+1}, \dots, X_n \text{ are } 0\}$$

$$\Big\downarrow$$

$$\mathbb{C}P^{n-i-1} = \{[0 : 0 : \cdots : 0 : X_{i+1} : \cdots : X_n]\}$$

$$H^{2n-2i}(N, \partial N) \overset{exc}{=} H^{2n-2i}(\mathbb{C}P^n, \overline{\mathbb{C}P^n - N})$$



$$= H^{2n-2i}(\mathbb{C}P^n, \mathbb{C}P^{n-i-1})$$

$$= H^{2n-2i}(\mathbb{C}P^n) = \mathbb{Z}$$

$$= H^{2n-2i}(\mathbb{C}P^{n-i})$$

$$= H^{2n-2i}(B^{2n-2i}, \partial B^{2n-2i})$$