Cup product, continued

$\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$

$\varphi \cup \psi \in C^{k+\ell}(X;R)$

$\varphi \cup \psi(\sigma) = \varphi(\sigma|[v_0,\dots,v_k]) \cdot \psi(\sigma|[v_k,\dots,v_{k+\ell}])$.

- $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$
- R associative $\Rightarrow$ $(\varphi \cup \psi) \cup \chi = \varphi \cup (\psi \cup \chi)$
- If R has $1 \in R$ then $[1] \in H^0(X;R)$ is a unit

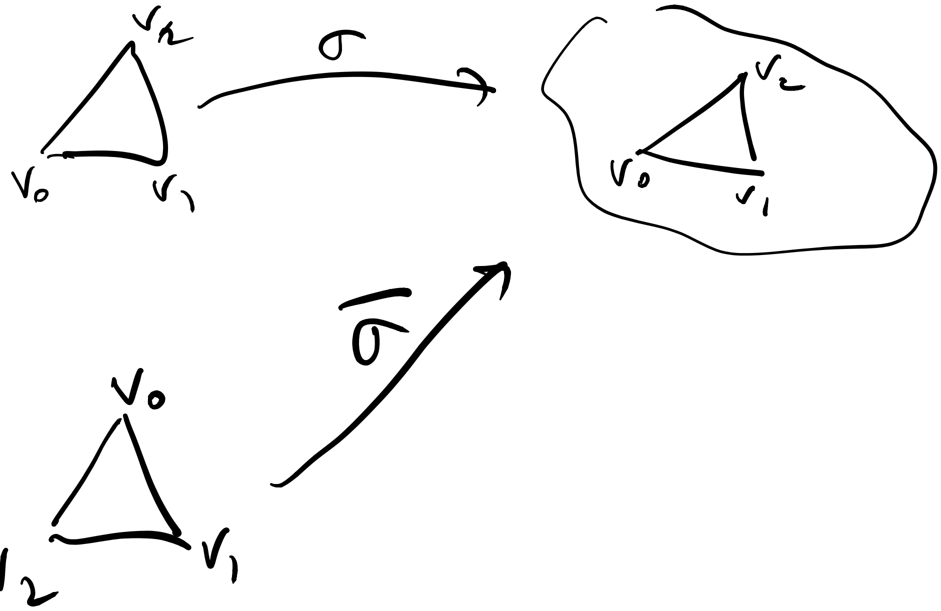
  $[1] \cup [\varphi] = [\varphi] \cup [1] = [\varphi]$

<u>Thm</u> R commutative $\Rightarrow$ $\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha$

if $\alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$

<u>Pf</u> Similar to the proof of the homotopy axiom and that subdivision induces $1\!\!1$ in H_*.

For $\sigma : \Delta^n \to X$ define $\bar{\sigma} : \Delta^n \to X$ to be composition of the affine homeo $\Delta^n \to \Delta^n$ that reverses the order of the vertices, $v_i \longmapsto v_{n-i}$, and σ.



Let $\varepsilon_n = (-1)^{\binom{n+1}{2}}$ parity of the reversal as a permutation.

Define $\rho : C_n(X) \to C_n(X)$

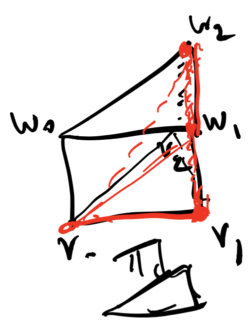
$$\rho(\sigma) = \varepsilon_n \bar\sigma.$$

Claim ρ is a chain morphism, $\rho\partial = \partial\rho$.

Claim ρ is chain homotopic to $\mathbb{1}$.

The chain homotopy is $P : C_n(X) \to C_{n+1}(X)$

$$P(\sigma) = \sum (-1)^i \varepsilon_{n-i} (\sigma\pi)\Big|[v_0,\dots,v_i,w_n,w_{n-1},\dots,w_i]$$



$$\partial P + P\partial = \rho - \mathbb{1}.$$

To finish:

$$(\rho^*\varphi \cup \rho^*\psi)(\sigma) = \varphi\left(\varepsilon_k\, \sigma|_{[v_k,\dots,v_0]}\right)\psi\left(\varepsilon_\ell\, \sigma|_{[v_{k+\ell},\dots,v_k]}\right)$$

$$\rho^*(\psi\cup\varphi)(\sigma) = \varepsilon_{k+\ell}\, \psi\left(\sigma|_{[v_{k+\ell},\dots,v_k]}\right)\cdot\varphi\left(\sigma|_{[v_k,\dots,v_0]}\right)$$

So $$\boxed{\varepsilon_k\varepsilon_\ell\, \rho^*\varphi\cup\rho^*\psi = \varepsilon_{k+\ell}\, \rho^*(\psi\cup\varphi)}$$

because R is commutative.

In H^* $\rho^* = 1$ and $\varepsilon_{k+\ell} = (-1)^{k\ell}\varepsilon_k\cdot\varepsilon_\ell$

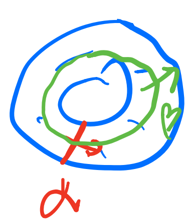
So $\alpha\cup\beta = (-1)^{k\ell}\beta\cup\alpha$. $\square$.

$$[\rho^*\varphi] = \underset{\substack{\| \\ \alpha}}{[\varphi]}$$

Relative versions

$$H^k(X)\times H^\ell(X,A) \longrightarrow H^{k+\ell}(X,A)$$

Ex.



$X = S^1\times I$
$A = \partial(S^1\times I)$

$\alpha\in H^1(X)$
$\beta\in H^1(X,A)$
$\alpha\cup\beta\in H^2(X,A)$

represented by a point.

$$\begin{array}{ccccc} H^1(X) & \times & H^1(X,A) & \xrightarrow{\cup} & H^2(X,A) \\ \| & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\ \alpha & & \beta & & \underline{\alpha \cup \beta} \end{array}$$

<u>More relative versions</u>

$$H^k(X,A) \times H^l(X,A) \longrightarrow H^{k+l}(X,A)$$

$$H^k(X,A) \times H^l(X,B) \longrightarrow H^{k+l}(X, A\cup B)$$

assuming NDR situation, e.g. CW complexes and subcomplexes.

$$C(X, A\cup B) \xleftarrow[\text{chain htpy equiv.}]{} C(X, A+B)$$

$f : X \to Y$

$f^* : H^*(Y;R) \to H^*(X;R)$

$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$ holds even on the cochain level.

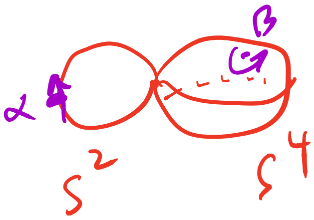
<u>Example</u> Cannot work out the cup product just from the cellular chain complex.

$S^2 \vee S^4$, $\mathbb{C}P^2$ $e^0 \cup e^2 \cup e^4$

Let $\alpha \in H^2 \cong \mathbb{Z}$ be a generator
$\beta \in H^4 \cong \mathbb{Z}$ generator.

Claim: In $S^2 \vee S^4$, $\alpha \cup \alpha = 0$

In $\mathbb{C}P^2$, $\alpha \cup \alpha = \pm\beta$ [postponed]



$\alpha \cup \alpha = 0$ because a point can be perturbed to be disjoint from itself.

Formal proof.

$$\begin{tikzcd} S^2 \arrow[r, hook, "i"] \arrow[rr, bend left, "\mathrm{id}"] & S^2 \vee S^4 \arrow[r, "\text{retraction } f"', "\text{collapse } S^4 \text{ to point}" below] & S^2 \end{tikzcd}$$

$f^*(\gamma) = \alpha$ $\qquad \gamma \in H^2(S^2) \cong \mathbb{Z}$

$\gamma \cup \gamma = 0 \in H^4(S^2) = 0$

$$\alpha \cup \alpha = f^*(\gamma) \cup f^*(\gamma) = f^*(\gamma \cup \gamma) = 0$$

Next goal $\quad H^*(\mathbb{C}P^n) = \bigoplus_{i=0}^{\infty} H^i(\mathbb{C}P^n) \cong \mathbb{Z}[\underset{\uparrow\ \text{deg } 2}{\alpha}]/\alpha^{n+1}=0,$

e.g. $H^*(\mathbb{C}P^2) = \mathbb{Z}[\alpha]/\alpha^3=0 = \langle \underset{0}{1}, \underset{2}{\alpha}, \underset{4}{\alpha^2} \rangle$

$$H^*(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[\underset{\text{deg } 1}{\alpha}]/\alpha^{n+1}=0 = \langle \underset{0}{1}, \underset{1}{\alpha}, \underset{2}{\alpha^2}, \dots, \underset{n}{\alpha^n} \rangle$$

Geometrically: $\alpha = [\mathbb{RP}^{n-1}]$

$$H^*(T^n) = \Lambda(\underset{\text{deg } 1}{\alpha_1, \dots, \alpha_n})$$

exterior algebra

$\alpha_i^2 = 0$

$\alpha_i \alpha_j = -\alpha_j \alpha_i$