$\boxed{\text{UCT}}$: (C_n) chain cx of free ab. groups.

$$0 \to \operatorname{Ext}(H_{n-1}(C), G) \to H^n(C;G) \to \operatorname{Hom}(H_n(C), G) \to 0$$

functorial, exact, splits non-functorially

$\boxed{\text{Pf}}$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Apply $\operatorname{Hom}(-, G)$, $A \rightsquigarrow A^*$

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 & Z_{n+1}^* \arrow[l] \arrow[u] & C_{n+1}^* \arrow[l] \arrow[u] & B_n^* \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & Z_n^* \arrow[l] \arrow[u, "0"] & C_n^* \arrow[l] \arrow[u, "\delta"] & B_{n-1}^* \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

Pass to LES in H^*

$$B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^*$$

$$0 \leftarrow \operatorname{Ker} i_n^* \leftarrow H^n(C;G) \leftarrow \operatorname{Coker} i_{n-1}^* \leftarrow 0$$

$$0 \to \underline{B_n \xrightarrow{i_n} Z_n} \to H_n(C) \to 0$$

$$0 \leftarrow \underset{1}{B_n^*} \xleftarrow{i_n^*} \underset{0}{Z_n^*} \leftarrow 0$$

$$\operatorname{Hom}(H_n(C),G) = \operatorname{Ker} i_n^*$$

$$\operatorname{Ext}(H_n(C),G) = \operatorname{Coker} i_n^*$$

This gives UCT.

<u>Splitting</u>: $\exists$ splitting $p : C_n \to Z_n$.

An element in $\operatorname{Hom}(H_n(C),G)$ is a homomorphism $Z_n \to G$ that vanishes on B_n.

Compose with p: Get a homo $C_n \to G$ that vanishes on B_n, so it's a cocycle!

$\boxtimes$

<u>Homology version</u>

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to \operatorname{Tor}(H_{n-1}(C),G) \to 0$$

Example Suppose $H_n(X)$ are f.g. ab. groups then

Then $H^n(X) \cong \text{Free part}(H_n(X)) \oplus \text{Torsion part}(H_{n-1}(X))$

$\mathbb{RP}^2$

	H_*	H^*
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}_2$	0
2	0	$\mathbb{Z}_2$

$H^1(X) = \text{Hom}(H_1(X), \mathbb{Z})$
always torsion free.

Example If $H_n(X)$ is f.g. then

$$H_n(X;\mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$$

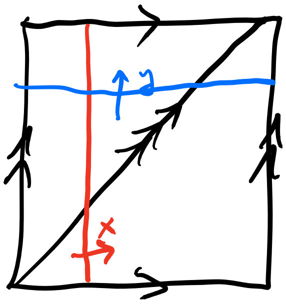
If $H_n(X) = \mathbb{Z}^k \oplus \text{torsion} \Rightarrow H_n(X;\mathbb{Q}) \cong \mathbb{Q}^k$.

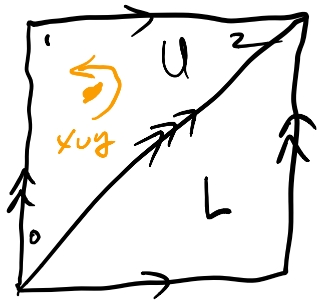
Cup Product $\quad \cup : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$

This turns $H^*(X) = \bigoplus_{i=0}^{\infty} H^i(X)$ into a <u>graded ring</u>.

This corresponds to "intersecting cocycles".

Ex.





$x \cup y$ assigns
-1 to U
0 to L

$y \cup x$

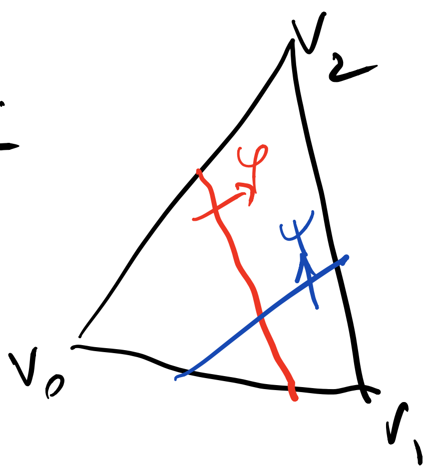
$y \cup x = -x \cup y$

Def. $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$. $\varphi \cup \psi \in C^{k+l}(X;R)$

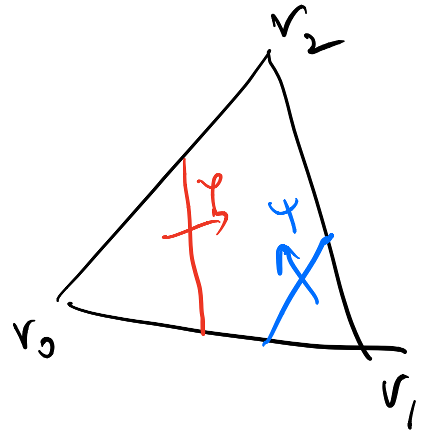
$$\varphi \cup \psi(\sigma) = \underset{\in R}{\varphi(\sigma|_{[v_0,\dots,v_k]})} \underset{\substack{\uparrow \\ \text{multiplication} \\ \text{in } R.}}{\cdot} \underset{\in R}{\psi(\sigma|_{[v_k,\dots,v_{k+l}]})}$$

(R is a ring, eventually commutative and with $1 \in R$)

Ex.



$$\varphi \cup \psi([v_0,v_1,v_2]) = \varphi([v_0,v_1]) \cdot \psi([v_1,v_2])$$
$$= 1 \cdot 1 = 1 \in \mathbb{Z}$$



$$\varphi \cup \psi([v_0, v_1, v_2]) = \varphi([v_0, v_1]) \cdot \psi([v_1, v_2]) \neq 0$$

Lemma $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

$k = \deg \varphi$

When interchanging two objects of degrees p, q, the signs $(-1)^{pq}$ pops up.

Cor. φ, ψ cocycles $\Rightarrow$ $\varphi \cup \psi$ cocycle

If in addition one is a coboundary $\Rightarrow$ $\varphi \cup \psi$ coboundary

So have a well defined product

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\cup} H^{k+l}(X;R)$$

$$[\varphi] \cup [\psi] = [\varphi \cup \psi]$$

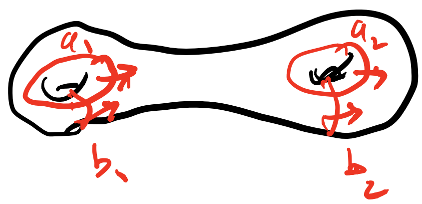
Thm S_g closed orientable surface of genus g. There is a basis $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ of $H^1(S_g) \cong \mathbb{Z}^{2g}$

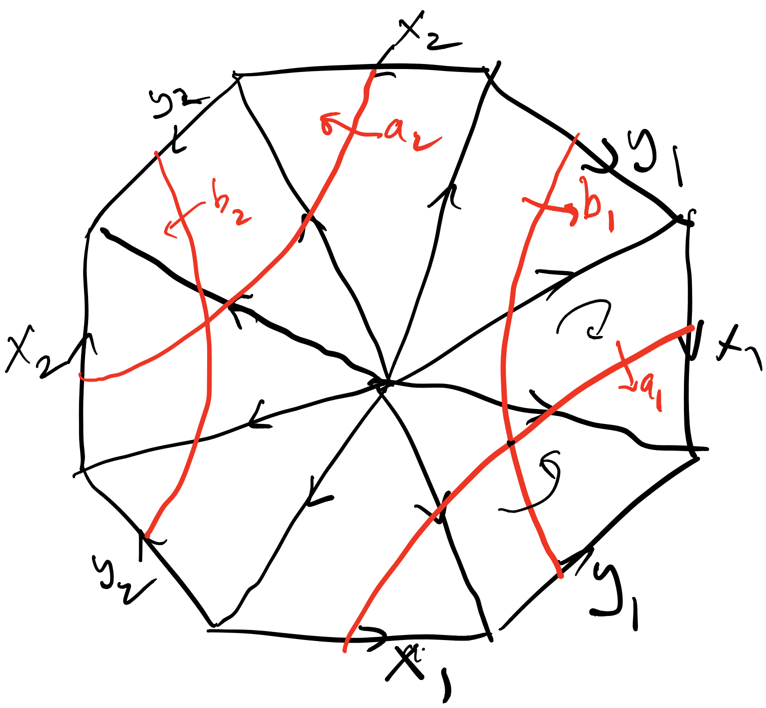
s.t. $a_i \cup a_j = b_i \cup b_j = 0$ $\quad \forall i, j$

$a_i \cup b_j = b_j \cup a_i = 0$ $\quad i \neq j$

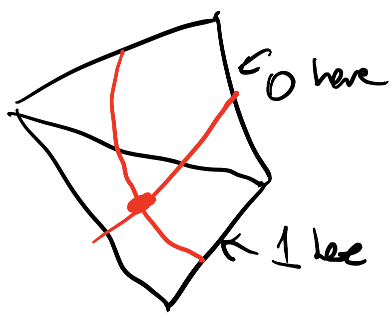
$a_i \cup b_i = - b_i \cup a_i$ is a fixed generator of $H^2(\Sigma_g) \cong \mathbb{Z}$ $\forall i$.

<u>$g=2$</u>

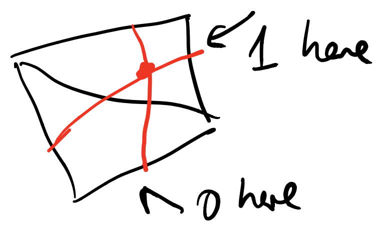




To compute $a_1 \cup b_1$



To compute $b_1 \cup a_1$



<u>Thm</u> R commutative $\Rightarrow$ $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$, $\alpha \in H^k(X;R)$, $\beta \in H^l(X;R)$