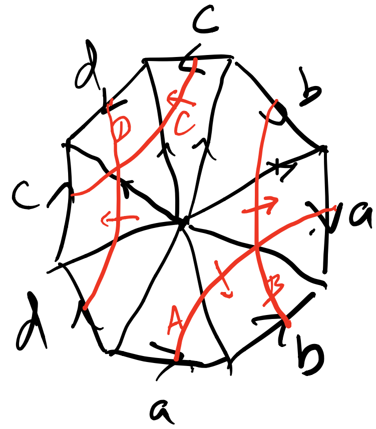
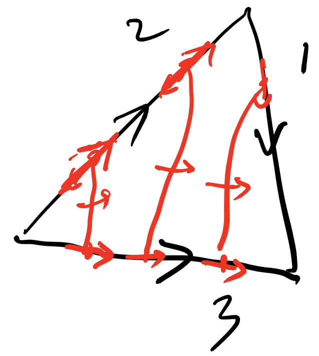
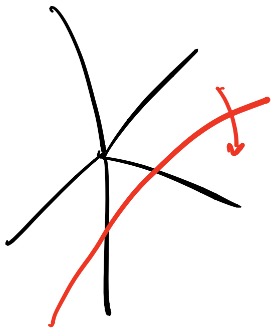
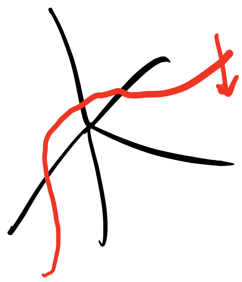
Ex.

A, B, C, D 1-cocycles.









Kronecker pairing

$$H^n(X) \otimes H_n(X) \xrightarrow{\langle\,,\,\rangle} \mathbb{Z}$$

$$\langle [\varphi], [z] \rangle = \varphi(z)$$

$$\langle [\varphi], [\partial w] \rangle = \varphi(\partial w) = \underbrace{\delta\varphi}_{0}(w) = 0$$

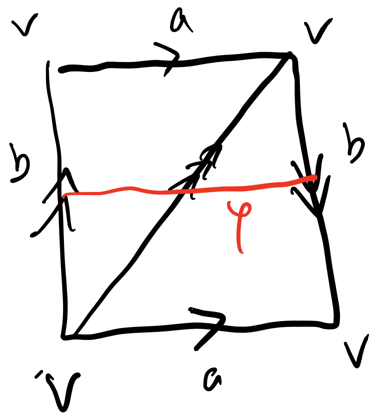
So $\langle\,,\,\rangle$ is independent of the choice of representating (co)cycles.

In above example $\langle A, a \rangle = 1$, $\langle A, b \rangle = 0$ etc

$\Rightarrow$ A, B, C, D are lin. independent in $H^1(X)$

a, b, c, d $\qquad\qquad$ $H_1(X)$.

<u>Ex.</u>



K = Klein bottle.

φ 1-cocycle over $\mathbb{Z}/2\mathbb{Z}$, it is <u>not</u> transversely orientable.

Kronecker pairing $\quad H^n(X;G) \otimes H_n(X;H) \to G \otimes H$

$[\varphi] \qquad [z] \longmapsto \sum h_i \cdot \underbrace{\varphi(\sigma_i)}_{\in G} \in G\otimes H$

$\quad\; z = \sum h_i \sigma_i$

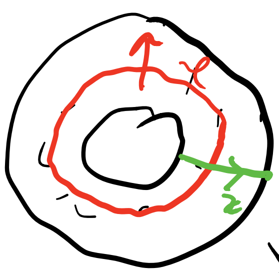
If R is a ring, have $R \otimes R \to R$

$$H^n(X;R) \otimes H_n(X;R) \to R$$

$$H^1(K;\mathbb{Z}/2\mathbb{Z}) \otimes H_1(K;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

$$\langle [\varphi], [b] \rangle = 1 \in \mathbb{Z}/2\mathbb{Z}$$

<u>Ex.</u> Relative cohomology $H^n(X,A)$ - here cocycles are pictures disjoint from A.



$[\varphi] \in H^1(X,A)$

$X = S^1 \times [0,1]$, $A = \partial X$

$$H^1(X,A)\otimes H_1(X,A)\longrightarrow\mathbb{Z}$$

$$\langle[\varphi],[z]\rangle = 1$$

Ex.

$H_1(\mathbb{R}P^3)=\mathbb{Z}/2\mathbb{Z}$

$H_2(\mathbb{R}P^3)=0$

$H_3(\mathbb{R}P^3)=\mathbb{Z}$

$H^1(\mathbb{R}P^3)=0$

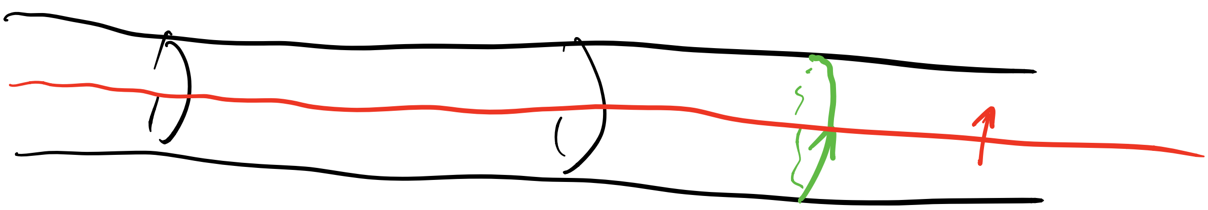
$H^2(\mathbb{R}P^3)=\mathbb{Z}/2\mathbb{Z}$

$H^3(\mathbb{R}P^3)=\mathbb{Z}$

H_i: corresponds to i-dim'l oriented submanifolds

H^i corresponds to transversely oriented codim i submanifolds

Ex.



$S^1\times\mathbb{R}$

Universal Coeff. thm

C chain complex of free ab. groups.

Then there is a functorial short exact sequence

$$0\longrightarrow \mathrm{Ext}(H_{n-1}(C),G)\longrightarrow H^n(C;G)\xrightarrow{k}\mathrm{Hom}(H_n(C),G)\to 0$$

which splits, but not functorially.

$\mathcal{K}$ = Kronecker pairing

$$[\varphi] \xmapsto{\mathcal{K}} \left([z] \longmapsto \varphi(z)\right)$$

Pf of Excision X, $\mathcal{U}$ = collection of subsets of X whose interiors cover X.

$$C^{\mathcal{U}}(X) \hookrightarrow C(X)$$

$\uparrow$ $\mathcal{U}$-small chain complex

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H^{\mathcal{U}}_{n-1}(X), G) \arrow[r] & H^n_{\mathcal{U}}(X, G) \arrow[r] & \mathrm{Hom}(H^{\mathcal{U}}_n(X), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X), G) \arrow[r] \arrow[u, "\cong"] & H^n(X, G) \arrow[r] \arrow[u] & \mathrm{Hom}(H_n(X), G) \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

so done by 5-lemma!

Review of Tor and Ext

If $A \to B \to C \to 0$ is exact seq. of ab. groups

then $A \otimes G \to B \otimes G \to C \otimes G \to 0$ is also exact, but $A \otimes G \to B \otimes G$ may not be injective even if $A \to B$ is.

Same for $\mathrm{Hom}(X, G) =: X^*$

$$A^* \leftarrow B^* \leftarrow C^* \leftarrow 0.$$

Idea: There a functors Tor & Ext so that if

$$0 \to A \to B \to C \to 0$$

then

$$0 \to \mathrm{Tor}(A,G) \to \mathrm{Tor}(B,G) \to \mathrm{Tor}(C,G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0$$

$$0 \leftarrow \mathrm{Ext}(A,G) \leftarrow \mathrm{Ext}(B,G) \leftarrow \mathrm{Ext}(C,G) \leftarrow A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$$

are exact

<u>Free resolutions</u> If H is an abelian group a <u>free resolution</u> of H is an exact seq.

$$\cdots \to F_2 \to F_1 \to F_0 \to H \to 0$$

where each F_i is a free abelian group.

Define $\mathrm{Tor}_i(H,G) = H_i\left(\cdots \to F_1\otimes G \to F_0\otimes G \to 0\right)$

$\mathrm{Ext}^i(H,G) = H^i\left(\cdots \leftarrow F_1^* \leftarrow F_0^* \leftarrow 0\right)$

$$\boxed{\begin{array}{l}\text{over } \mathbb{Z},\ \mathrm{Ext}^i, \mathrm{Tor}_i \equiv 0,\ i>1, \\ \mathrm{Ext} := \mathrm{Ext}^1,\ \mathrm{Tor} := \mathrm{Tor}_1.\end{array}}$$

<u>Ex.</u> $\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2)$

$$0 \to \underset{F_1}{\mathbb{Z}} \xrightarrow{\times 2} \underset{F_0}{\mathbb{Z}} \to \mathbb{Z}_2 \to 0$$

$$\rightsquigarrow \mathrm{Hom}(-,\mathbb{Z}_2)$$

$$0 \leftarrow \mathbb{Z}_2 \xleftarrow[=0]{\times 2} \mathbb{Z}_2 \leftarrow 0$$

0th cohomology

$\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2$, $\mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2$

$$\left.\begin{array}{l}\mathrm{Tor}_0(H,G) = H\otimes G \\ \mathrm{Ext}^0(H,G) = \mathrm{Hom}(H,G)\end{array}\right\} \text{by right exactness}$$

<u>Independence of the resolution</u>

Universal property (of projective modules)

$$\begin{tikzcd} & F \arrow[dl, dotted, "\tilde{\alpha}"'] \arrow[d, "\alpha"] & \\ A \arrow[r, "\pi"] & B \arrow[r] & 0 \end{tikzcd}$$

Claim Any two free (or projective) resolutions of H are chain homotopy equivalent. ($\Rightarrow$ independence)

$$\begin{array}{ccccccccc}
\cdots & \to & F_1 & \xrightarrow{\partial} & F_0 & \to & H & \to & 0 \\
 & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \| & & \\
\cdots & \to & F_1' & \xrightarrow{\partial'} & F_0' & \xrightarrow{\partial''} & H & \to & 0
\end{array}$$

Need to see that $\mathrm{im}(\varphi_0 \partial) \subseteq \mathrm{im}\,\partial' = \ker \partial''$

Continue inductively.

Construct $\varphi_i' : F_i' \to F_i$ in the same way.

Argue $\varphi_i'\varphi_i \simeq 1$, $\varphi_i\varphi_i' \simeq 1$ in a similar way.

Calculations

- $\mathrm{Tor}(H_1 \oplus H_2, G) = \mathrm{Tor}(H_1, G) \oplus \mathrm{Tor}(H_2, G)$
- $\mathrm{Tor}(\mathbb{Z}, G) = 0$ $\qquad 0 \to \underset{F_0}{\overset{\shortparallel}{\mathbb{Z}}} \to \mathbb{Z} \to 0$
- $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker(G \xrightarrow{\times n} G)$, $\quad \mathbb{Z}_n \otimes G = G/nG$.

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

$$\Big\downarrow \otimes G$$

$$0 \to G \xrightarrow{\times n} G \to 0$$

$$\mathrm{Tor}(\mathbb{Z}_n,\mathbb{Z}_m) = \mathbb{Z}_{\gcd(n,m)}\,, \qquad \mathbb{Z}_n \otimes \mathbb{Z}_m = \mathbb{Z}_{\gcd(n,m)}$$

$$\mathrm{Ext}(H_1 \oplus H_2, G) = \mathrm{Ext}(H_1, G) \oplus \mathrm{Ext}(H_2, G)$$

$$\mathrm{Ext}(\mathbb{Z}, G) = 0$$

$$\mathrm{Ext}(\mathbb{Z}_n, G) = G/nG$$

$$\mathrm{Hom}(\mathbb{Z}_n, G) = \ker(G \xrightarrow{\times n} G)$$

$$\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(n,m)}\,.$$

$$0 \to A \to \,?\, \to B \to 0$$

\# ways of filling in ? $\underset{1-1}{\longleftrightarrow}$ $\mathrm{Ext}(B,A)$

$$0 \to \mathbb{Z}_2 \to \,?\, \to \mathbb{Z}_2 \to 0$$

$$\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2 \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathbb{Z}_4 \end{cases}$$

<u>LES</u>: $0 \to A \to B \to C \to 0$.

Build resolutions:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

<u>$B_i = A_i \oplus C_i$</u>

A_i, C_i arbitrary

Apply $\otimes G$, remove bottom row. Get LES in homology

$$\cdots \to \mathrm{Tor}(B,G) \to \mathrm{Tor}(C,G) \to A \otimes G \to B \otimes G \to \underset{\substack{\uparrow \\ 0^{\text{th}} \text{ homology of } C \otimes G}}{C \otimes G} \to 0$$