Thm $X, Y \subset \mathbb{R}^n$ compact, homeomorphic.

S^n

Then $H_i(\mathbb{R}^n - X) \cong H_i(\mathbb{R}^n - Y)$.

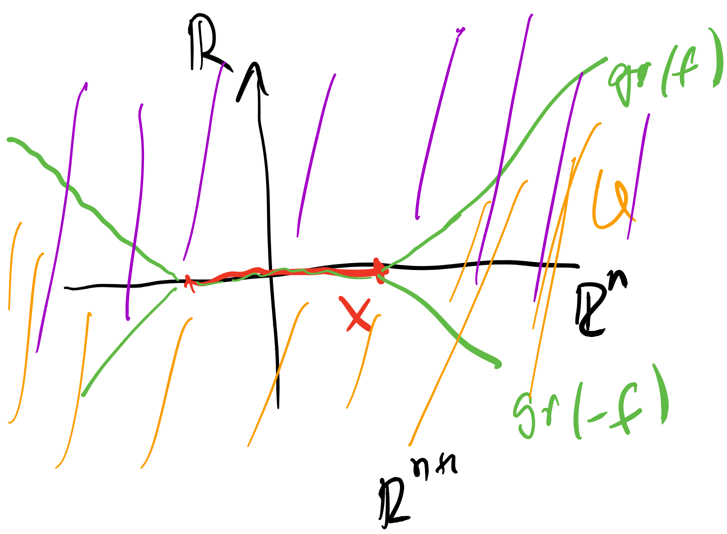
S^n $\qquad$ S^n

Ex. $\mathbb{R}^3$, knots

2 steps in the proof.

① $X \subset \mathbb{R}^n = \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$

Claim $\tilde{H}_{i+1}(\mathbb{R}^{n+1} - X) \cong \tilde{H}_i(\mathbb{R}^n - X)$



$f : \mathbb{R}^n \to [0, \infty)$
"distance from X"

$f(z) = d(z, X)$
$= \inf_{x \in X} d(z, x)$

U = region below gr(f)

V = region above gr(−f)

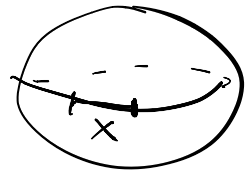
U, V contractible

$U \cup V = \mathbb{R}^{n+1} - X$.

$U \cap V = \mathbb{R}^n - X$.

Claim follows from M-V.

[similar for $X \subset S^n \subset S^{n+1}$



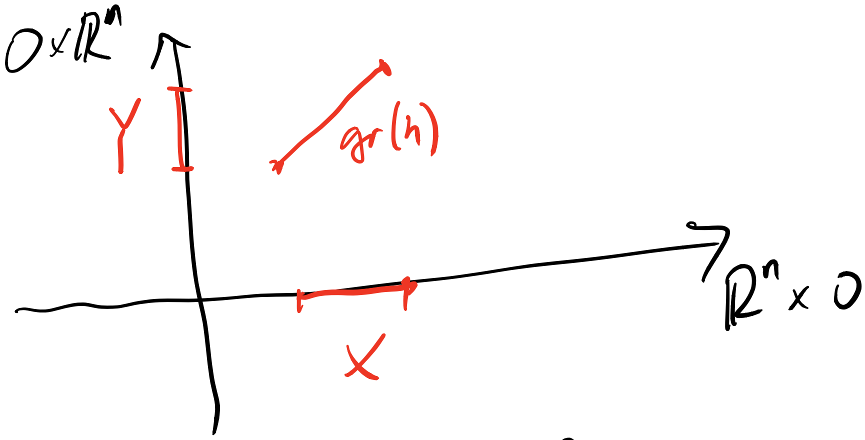
]

② "Klee trick".

Using step 1 we can assume

$X \subset \mathbb{R}^n \times 0 \subset \mathbb{R}^{2n}$

$Y \subset 0 \times \mathbb{R}^n \subset \mathbb{R}^{2n}$



$h : X \to Y$ homeo.

<u>Claim</u> $\exists$ homeo $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that takes X to $gr(h)$, and another homeo that takes Y to $gr(h)$.

$\rightsquigarrow$ so X, Y have homeomorphic complements, so Thm holds!

Pf of Claim: By Tietze, h extends to a continuous function $\tilde{h}: \mathbb{R}^n \to \mathbb{R}^n$

Define
$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$
$$(a,b) \mapsto (a, b + \tilde{h}(a))$$

homeo with inverse $(a,b) \mapsto (a, b - \tilde{h}(a))$.

Takes X to $\operatorname{gr}(h)$. $\square$

Cor: If $X \subset \mathbb{R}^n$, $X \simeq D^n$ then $\mathbb{R}^n - X$ is connected, $n \geq 2$, has 2 components, $n = 1$.

If $X \simeq S^{n-1}$, then $\mathbb{R}^n - X$ has 2 components, $n \geq 2$, 3 components if $n = 1$.

Ex. If $X \subset S^n$, $X \simeq D^k$ then $\tilde{H}_i(S^n - X) = $ triv.

If $X \subset S^n$, $X \simeq S^k$ then $\tilde{H}_i(S^n - X) = \begin{cases} \mathbb{Z}, & i = n-k-1 \\ 0, & \text{otherwise.} \end{cases}$

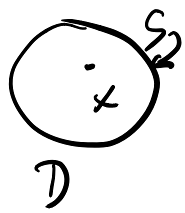
Invariance of Domain

If $X, U \subseteq \mathbb{R}^n$, $X \underset{\text{homeo}}{\simeq} U$, U open $\Rightarrow$ X open.

Pf: $x \in X$, need to show $x \in \operatorname{int} X$.

From assumption $\exists\, D \subseteq X$, $(D, x) \approx (D^n, 0)$

Let $S \subseteq D$ corresp. to ∂D^n under this homeo.



$n>1$

$$\underbrace{\mathbb{R}^n - S}_{\text{has 2 components}} = (\underbrace{\mathbb{R}^n - D}_{\text{connected}}) \cup (\underbrace{D - S}_{\approx \mathring{D}^n,\ \text{connected}})$$

$\Rightarrow$ The 2 components of $\mathbb{R}^n - S$ are $\mathbb{R}^n - D$ and $D - S$ (and they are both open since $\mathbb{R}^n - S$ is loc. connected).

So $D - S$ is open. So $x \in \operatorname{int} D \subseteq \operatorname{int} X$.

$n=1$ Same pf except now LHS has 3 components, $\mathbb{R}^n - D$ has 2 components, $D - S$ is connected. $\square$

Cohomology

G ab. group.

$$C^n(X;G) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X);G) \quad \text{cochain cx}$$

$$= \{\text{functions } \{\text{singular } n\text{-simplices } \Delta^n \to X\} \to G\}$$

$$0 \to C^0(X;G) \xrightarrow{\delta} C^1(X;G) \xrightarrow{\delta} C^2(X;G) \to \cdots$$

$$\delta\varphi(\sigma) = \varphi(\partial\sigma)$$

<u>Def</u> $$H^n(X;G) = \frac{\mathrm{Ker}(\delta : C^n \to C^{n+1})}{\mathrm{Im}(\delta : C^{n-1} \to C^n)}$$

n^{th} cohomology with coeffs in G.

<u>Properties</u>

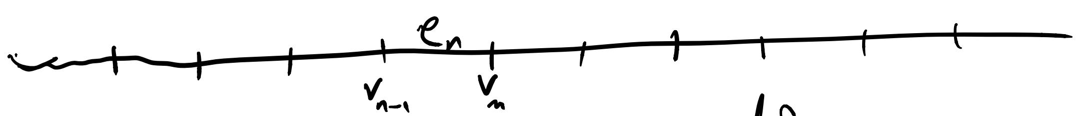
- Contravariant functor $f : X \to Y \rightsquigarrow f^* : H^n(Y;G) \to H^n(X;G)$
- homotopy $f \simeq g : X \to Y \Rightarrow f^\# \simeq g^\# \Rightarrow f^* = g^*$.
- excision. Now the proof doesn't work because H^n is <u>not</u> compactly supported.
- equivalence of Δ-cohomology, singular, cellular cohomology.

- if G is a field, then $H^n(X;G)$ is a vector space over G.
- Long exact sequences (X, A)

$$\cdots \to H^n(X) \to H^n(A) \to H^{n+1}(X,A) \to H^{n+1}(X) \to \cdots$$

- Mayer–Vietoris.

Ex.



$H^0_\Delta(\mathbb{R})$ 0-cochain φ: vertices $\to \mathbb{Z}$

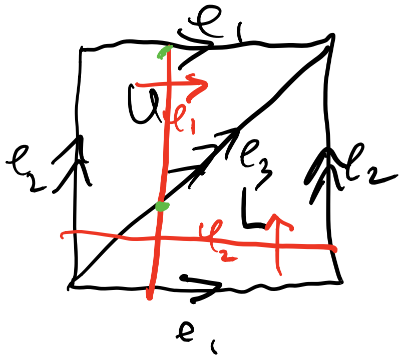
$$\delta\varphi(e_n) = \varphi(\partial e_n) = \varphi(v_n) - \varphi(v_{n-1})$$

φ cocycle $\Leftrightarrow$ $\varphi = $ const.

$$H^0(\mathbb{R}) = \mathbb{Z} \qquad \left(H^0(\mathbb{R}; G) = G \right)$$

Ex. $H^n(pt; G) = \begin{cases} G, & n = 0 \\ 0, & n \neq 0. \end{cases}$

Ex.



$C^1 = $ functions $\{e_1, e_2, e_3\} \to \mathbb{Z}$.

$$\delta\varphi(U) = \varphi(\partial U) = \varphi(e_1) + \varphi(e_2) - \varphi(e_3)$$

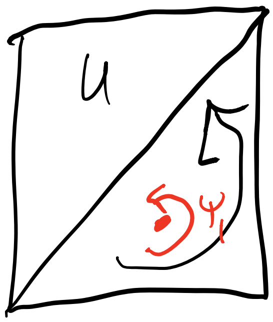
$$\delta\varphi(L) = \varphi(e_1) + \varphi(e_2) - \varphi(e_3).$$

So $Z^1 \cong \mathbb{Z}^2 = \langle \varphi_1, \varphi_2 \rangle$, $\quad \varphi_1 : e_1, e_3 \mapsto 1$, $e_2 \mapsto 0$

$\varphi_2 : e_2, e_3 \mapsto 1$, $e_1 \mapsto 0$

$H^1(T^2) \cong \mathbb{Z}^2$

$H^2(T^2) = ?$



$U \mapsto 0$

$L \mapsto 1$

$\delta\varphi(U) = \varphi(\partial U)$

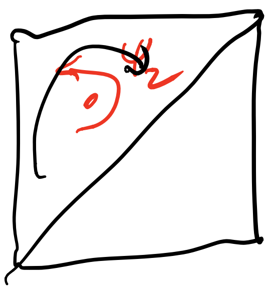
$= \varphi(e_1 + e_2 - e_3)$

$= -1$

$\psi_1 - \psi_2 = -\delta\varphi$

$\varphi = +1 \; -1$ on e_3

0 on e_1, e_2



$\delta\varphi(L) = \varphi(\partial L) = \varphi(e_1 + e_2 - e_3)$

$= -1$

$U \mapsto -1$

$L \mapsto 0$

$H^2(T^2) \cong \mathbb{Z}$