Cellular homology continued

X CW complex

$$\cdots \to W_2 \xrightarrow{\partial} W_1 \xrightarrow{\partial} W_0 \to 0$$

$$W_i = \bigoplus_{e^i_\alpha} \mathbb{Z}$$

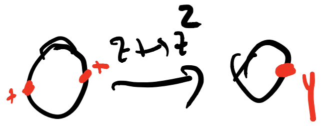
$$\partial e^i_\alpha = \sum_\beta d_{\alpha\beta} e^{i-1}_\beta, \qquad d_{\alpha\beta} = \text{algebraic count of } f_\alpha^{-1}(y)$$

$$y \in e^{i-1}_\beta, \quad f_\alpha : S^{i-1} \to X^{i-1}$$

Ex. $\mathbb{R}P^n = e_0 \cup e_1 \cup \cdots \cup e_n$

$f : S^{k-1} \to \mathbb{R}P^{k-1}$ double cover.
att. map for e_k.

$$0 \to \underset{n}{\mathbb{Z}} \to \cdots \underset{2}{\mathbb{Z}} \xrightarrow[2]{\partial_2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to 0$$



$S^2 \to \mathbb{R}P^2$

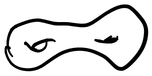
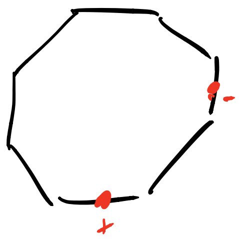
$$0 \to \mathbb{Z} \to \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

n even

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/2\mathbb{Z}, & i \text{ odd}, \ 0 < i < n \\ 0, & \text{otherwise.} \end{cases}$$

n odd $\qquad H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & i=0,n \\ \mathbb{Z}/2\mathbb{Z}, & i \text{ odd}, \ 0<i<n \\ 0 & \text{otherwise}. \end{cases}$

Similarly, compute $H_i(\mathbb{RP}^\infty)$

<u>Ex.</u>  

$$\begin{array}{ccccccccc} & & 2 & & 1 & & 0 & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}^4 & \xrightarrow{0} & \mathbb{Z} & \to & 0 \end{array}$$

$\Rightarrow H_1 = \mathbb{Z}^4$

$H_2 = \mathbb{Z}$

$f: X^{CW} \to Y^{CW}$ cellular, $f(X^n) \subseteq Y^n$

induces a chain homomorphism $\underbrace{H_n(X^n, X^{n-1})}_{= W_n^X} \to \underbrace{H_n(Y^n, Y^{n-1})}_{= W_n^Y}$

<u>Fact</u> This induces f_* in homology

Homology with coefficients

G abelian group $(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z})$

$$C_n(X;G) = \bigoplus_{\sigma:\Delta^n \to X} G = \Big\{ \underbrace{\sum g_i \sigma_i}_{\text{finite sum}} \ \Big| \ g_i \in G, \ \sigma_i : \Delta^n \to X \Big\}$$

$$C_n(X;\mathbb{Z}) = C_n(X)$$

$$C_n(X;G) = C_n(X) \otimes_{\mathbb{Z}} G$$

$$\cdots \xrightarrow{\partial^G} C_1(X;G) \xrightarrow{\partial^G} C_0(X;G) \to 0$$

$$\partial^G = \partial^{\mathbb{Z}} \otimes \mathbb{1}_G$$

$$\partial^G(\sigma) = \sum (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]$$

<u>Def</u>. $H_n(X;G) = n^{th}$ homology of this complex.

Everything generalizes:

1) It's a functor

2) If G is a field then $H_n(X;G)$ is a vector space over G.

3) Homotopy axiom $f \simeq g : X \to Y \Rightarrow f_* = g_* : H_n(X;G) \to H_n(Y;G)$

$$f_\# - g_\# = P\partial + \partial P \quad \text{in } C_n(X) \to C_n(Y)$$

$$\otimes G \rightsquigarrow \quad f^G_\# - g^G_\# = P^G \partial^G + \partial^G P^G.$$

4) Excision $S : C_n(X) \to C_n(X)$ is chain homotopic to $\mathbb{1}$ $\Rightarrow$ $S^G : C_n(X;G) \to C_n(X;G)$ is chain homotopic to $\mathbb{1}$.

+ homology has compact supports

$C_n^{\alpha}(X) \hookrightarrow C_n(X)$ induces iso in homology.

5) Long exact sequences.

(X, A) $\qquad 0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$

<u>Caution</u> $\otimes G$ does not always preserve short exact sequences.

$$0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0 \qquad \otimes \mathbb{Z}/3\mathbb{Z}$$

$$0 \to \underline{\mathbb{Z}/3\mathbb{Z}} \xrightarrow{0} \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

But here it remains exact!

Split exact sequences remain exact after $\otimes G$.

$$0 \to C_n(A; G) \to C_n(X; G) \to C_n(X, A; G) \to 0$$

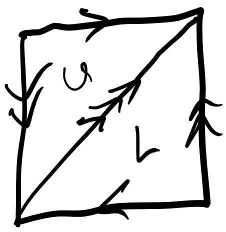
still exact,

$\rightsquigarrow$ induces LES.

6) Mayer-Vietoris

7) Δ-homology, cellular homology

8) $H_n(pt; G) = \begin{cases} G, & n=0 \\ 0, & \text{otherwise.} \end{cases}$

Ex. 

$\mathbb{RP}^2$

$H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

$\partial(U+L) = 0$!

Ex. $H_i(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\otimes \mathbb{Z}/2\mathbb{Z}$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0$$

$\partial^{\mathbb{Z}/2\mathbb{Z}} \equiv 0.$

$$H_i(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & i = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

Surprising fact: If you know $H_n(X)$ for all n, then you can compute $H_n(X; G)$ for all n.

Sketch why this is true

$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$, chain cx,

C_i f.g. free abelian

Fact: This chain complex breaks up into a direct sum of "very short" chain complexes

$$\cdots \to 0 \to \overset{i}{\mathbb{Z}} \to 0 \to \cdots$$

$$\cdots \to \overset{i}{\mathbb{Z}} \xrightarrow{\times k} \overset{i-1}{\mathbb{Z}} \to 0 \to \cdots, \quad k \geq 1.$$

$H_i(\ ;\mathbb{Z}) = \mathbb{Z}$

$H_i(\ ;G) = G$

$H_i(\ ;\mathbb{Z}) = 0$, $H_{i-1}(\ ;\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$

$H_i(\ ;G) = \ker(g \mapsto kg) = \mathrm{Tor}(\mathbb{Z}/k\mathbb{Z}, G)$

$H_{i-1}(\ ;G) = \mathrm{Coker}(g \mapsto kg) = \mathbb{Z}/k\mathbb{Z} \otimes G$

$$\boxed{H_i(X;G) \cong H_i(X) \otimes G \oplus \mathrm{Tor}(H_{i-1}(X);G)}$$

E.g. $G = \mathbb{Q}$

$H_i(X;\mathbb{Q}) = H_i(X) \otimes \mathbb{Q}$ (Tor $= 0$ for fields)

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[l, bend right] \arrow[d] & {} \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}
$$