Degree

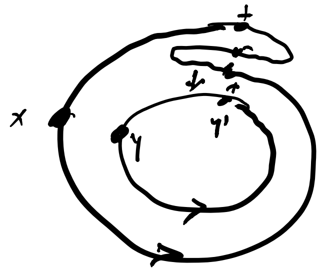
$f: S^n \to S^n$ smooth

$y \in S^n$ reg. value

$f^{-1}(y) = \{x_1, \dots, x_k\}$

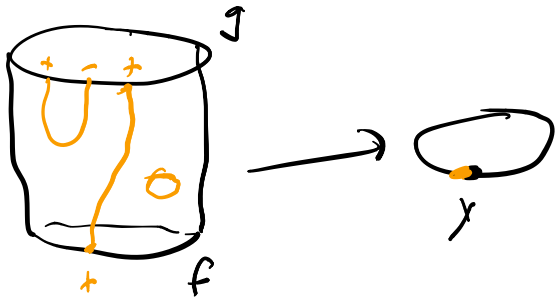
$df_{x_i}: T_{x_i} S^n \to T_y S^n$

$\varepsilon_i = \operatorname{sgn} |df_{x_i}|$



$\deg f = \sum_i \varepsilon_i$

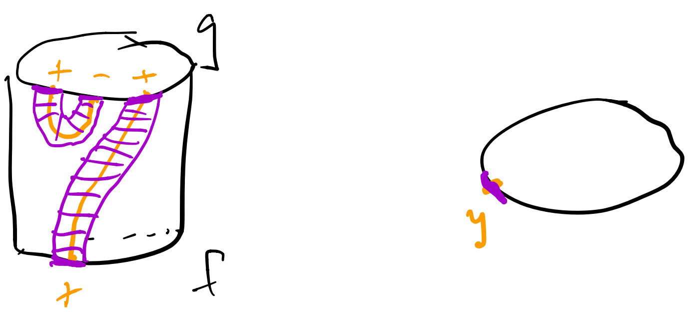
$f \simeq g$ smoothly $\Rightarrow \deg f = \deg g$



If $f: S^n \to S^n$ is continuous,

then $f \simeq f'$ smooth. Def $\deg f = \deg f'$.

Hopf degree theorem $[S^n, S^n] \xrightarrow{\deg} \mathbb{Z}$ is a bijection.



Properties

$\deg \mathbb{1} = 1$

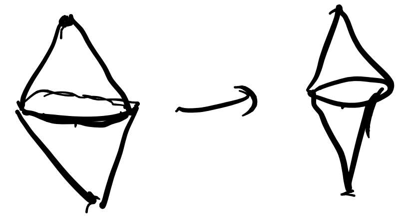
f not surj. $\Rightarrow \deg f = 0$

$f \simeq g \Leftrightarrow \deg f = \deg g$

f reflection in hyperplane $\Rightarrow \deg f = -1$

f antipodal ($f = -\mathbb{1}$) $\Rightarrow \deg f = (-1)^{n+1}$.

$\deg Sf = \deg f$



Prop If $\deg f = k$ then $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by k.

Pf. For $n=1$ last time.

For $n>1$ by Hopf $f \simeq Sg$, $g : S^{n-1} \to S^{n-1}$ has degree k.

$$\underset{\times k}{\circlearrowright} H_n(S^n) \underset{S}{\cong} H_{n-1}(S^{n-1}) \circlearrowleft \times k$$

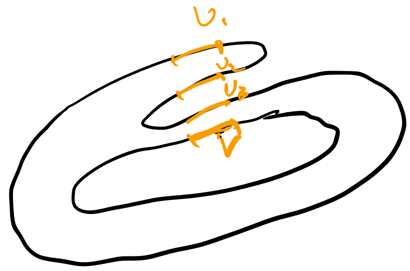
$Sg \simeq f$

Cor. $\deg fg = \deg f \cdot \deg g$

Aside Orientation of S^n - choice of a generator of $H_n(S^n) \cong \mathbb{Z}$

Local orientation at $x \in S^n$ - choice of a generator of $H_n(S^n, S^n - x) \cong \mathbb{Z}$.

$H_n(S^n) \xrightarrow{\cong} H_n(S^n, S^n - x)$ global orientation determines local $\forall x$.



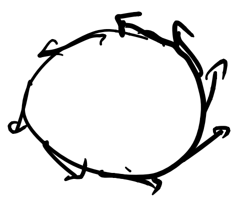
$$f : (U_i, U_i - x_i) \xrightarrow{\cong} (V, V - y)$$

$$\varepsilon_i = \begin{cases} +1, & f_* \text{ sends loc. orient. to loc. orient.} \\ -1 & \text{otherwise} \end{cases}$$

Hatcher: $\deg f = \sum \varepsilon_i$

Applications

1. S^n admits a continuous nonvanishing vector field $\Leftrightarrow n$ odd.



$\boxed{\Leftarrow}$ $V(x_0, x_1, \dots, x_n) = (-x_1, x_0, -x_3, x_2, \dots)$

$\boxed{\Rightarrow}$ $\mathbb{1} \simeq -\mathbb{1}$

$$h_t(x) = (\cos t)x + (\sin t)\cdot V(x)$$

$$1 = (-1)^{n+1} \Rightarrow n \text{ odd}.$$

2. If $f : S^n \to S^n$ does not fixed pts, then $f \simeq -\mathbb{1}$

<u>Pf.</u> $f(x)$ and $-x$ are never antipodal, so use straight line homotopy followed by radial projection

3. If $G \curvearrowright S^n$ freely, n even $\Rightarrow G = 1$ or $\mathbb{Z}/2\mathbb{Z}$.

<u>Pf.</u> $\deg : G \to \{1, -1\}$ homomorphism

Any $g \in$ kernel has deg 1, hence has fixed pts.

So kernel is trivial.

<u>Cellular homology</u>

X CW complex

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1}) \cong \begin{cases} 0, & k \neq n \\ \bigoplus_{\substack{n\text{-cells} \\ \text{of } X}} \mathbb{Z}, & k = n \end{cases}$$

$$\bigvee_{\substack{n\text{-cells} \\ \text{of } X}} S^n$$

Facts $H_k(X^n) = 0$, $k > n$

$H_k(X^n) \xrightarrow{\cong} H_k(X)$, $n > k$

Cellular chain complex

$$\cdots \to H_2(X^2, X^1) \xrightarrow{d} H_1(X^1, X^0) \xrightarrow{d} H_0(X^0) \to 0$$

induced by ∂

$$[z] \longmapsto [\partial z]$$

Thm The homology of this complex is $\cong H_*(X)$.

Pf.

$$\begin{tikzcd}
 & & 0 \arrow[d] & H_n(X^{n+1}) \cong H_n(X) \arrow[r] & 0 & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[rr, "d"] & & H_n(X^n, X^{n-1}) \arrow[dr, "\partial'"] \arrow[rr, "d'"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j'"] & \\
 & & & 0 \arrow[ur] & &
\end{tikzcd}$$

$$H_n(X) = H_n(X^n) / \operatorname{im} \partial$$

Claim $\operatorname{Im} j = \ker d'$

$\subseteq$ triangle commutes

$\supseteq$ $\ker d' = \ker \partial'$ since j' is injective,

<u>Fact</u> differential can be computed using degrees:

$$
\begin{array}{ccc}
H_n(X^n, X^{n-1}) & \xrightarrow{\ d\ } & H_{n-1}(X^{n-1}, X^{n-2}) \\
\rotatebox{90}{$\cong$} & & \rotatebox{90}{$\cong$} \\
\bigoplus_{e^n_\alpha} \mathbb{Z} & & \bigoplus_{e^{n-1}_\beta} \mathbb{Z}
\end{array}
$$

$$d\, e^n_\alpha = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$$

$$
\begin{tikzcd}
S^{n-1}_\alpha \arrow[r, "f_\alpha", "\substack{\text{att. map} \\ \text{for } e^n_\alpha}"'] \arrow[rrr, bend right, "d_{\alpha\beta} = \text{degree of this map}"'] & X^{n-1} \arrow[r] & X^{n-1}/X^{n-2} = \bigvee_{e^{n-1}_\beta} S^{n-1} \arrow[r, "\text{collapse}", "\substack{\text{all} \\ S^{n-1}_\gamma,\ \gamma \neq \beta}"'] & S^{n-1}_\beta
\end{tikzcd}
$$