(7) Equivalence of Δ-homologies and singular homology

X Δ-complex

$$\underset{\text{simplicial chain cx}}{\Delta_n(X)} \subset \overset{\text{singular ch. complex}}{C_n(X)}$$

Thm Inclusion induces isomorphism

$$H_n^{\Delta}(X) \xrightarrow{\cong} H_n(X)$$

The same holds for pairs.

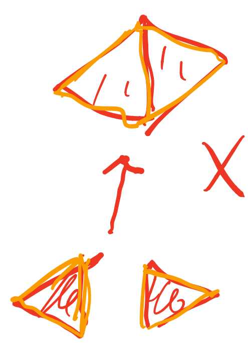
Pf. (X,A) pair of Δ-complexes.

Case 1 $(X,A) = (\Delta^n, \partial\Delta^n)$

True by application (1).

Case 2 $\dim X = k$, $A = X^{(k-1)}$.

$$\underset{\substack{\text{simplices}\\ \text{in } X-A}}{\coprod} (\Delta^k, \partial\Delta^k) \longrightarrow (X, A)$$



(iso) in both homologies:

- in H^{Δ} by definition
- in H by excision
- so done by Case 1.

Case 3 $\dim X < \infty$, $A = \phi$.

Induction on dimension k. If $k = 0$ ✓

Say $\dim X = k$

$$\begin{tikzcd}
H^\Delta_{i+1}(X, X^{k-1}) \arrow[r] \arrow[d, "\cong"', "\text{case2}"] & H^\Delta_i(X^{k-1}) \arrow[r] \arrow[d, "\cong", "\text{ind.}"'] & H^\Delta_i(X) \arrow[r] \arrow[d, "\cong"] & H^\Delta_i(X, X^{k-1}) \arrow[r] \arrow[d, "\cong", "\text{Case2}"'] & H^\Delta_{i-1}(X^{k-1}) \arrow[d, "\cong", "\text{ind.}"'] \\
H_{i+1}(X, X^{k-1}) \arrow[r] & H_i(X^{k-1}) \arrow[r] & H_i(X) \arrow[r] & H_i(X, X^{k-1}) \arrow[r] & H_{i-1}(X^{k-1})
\end{tikzcd}$$

naturality $\Rightarrow$ all squares commute.

5 lemma $\Rightarrow$ ✓

Case 4 $\dim X = \infty$, $A = \phi$.

homology has compact supports

onto: $H^\Delta_n(X) \to H_n(X)$

$[z]$, $z = \sum n_i \sigma_i$, support$(z) \subset$ finite subcomplex

So Case 3 $\Rightarrow$ onto.

1-1 similar. "CW-yoga + compact supports"

Case 5 General pairs (X, A). 5-lemma. □

⑧ Mayer-Vietoris

$X = A \cup B$, $\text{int} A \cup \text{int} B = X$. Then $\exists$ LES

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial^{MV}} H_{n-1}(A \cap B) \to \cdots$$

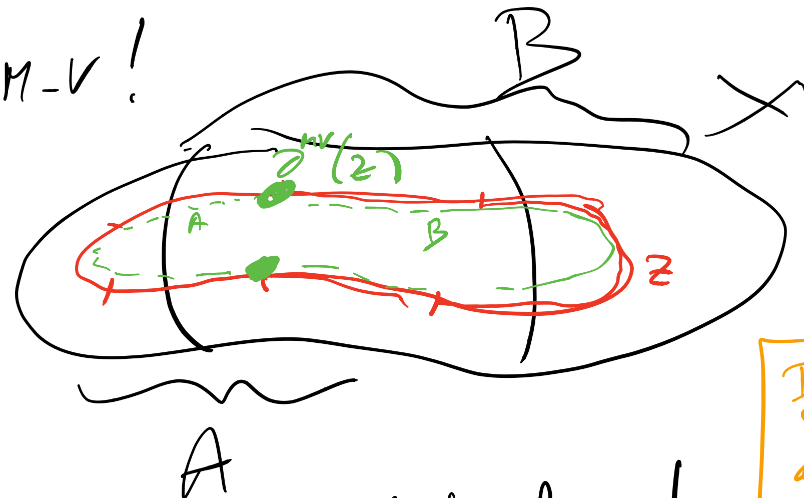
Pf. $0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(A+B) \to 0$

Ψ: $x \longmapsto (x, -x)$

$(x, y) \longmapsto x + y$

$C_n(A+B) \simeq_{\text{homology iso}} C_n(X)$

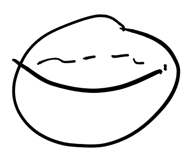
LES $\Rightarrow$ M-V!



If (X, A) and (X, B) are NDR then $\text{int} A \cup \text{int} B = X$ can be discarded.

Remark: Works with reduced homology!

Ex. $S^n = D^n_+ \cup D^n_-$

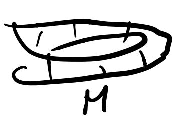


$$\tilde{H}_i(D^n_+) \oplus \tilde{H}_i(D^n_-) \to \tilde{H}_i(S^n) \xrightarrow{\cong} \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(D^n_+) \oplus \tilde{H}_{i-1}(D^n_-)$$

$\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

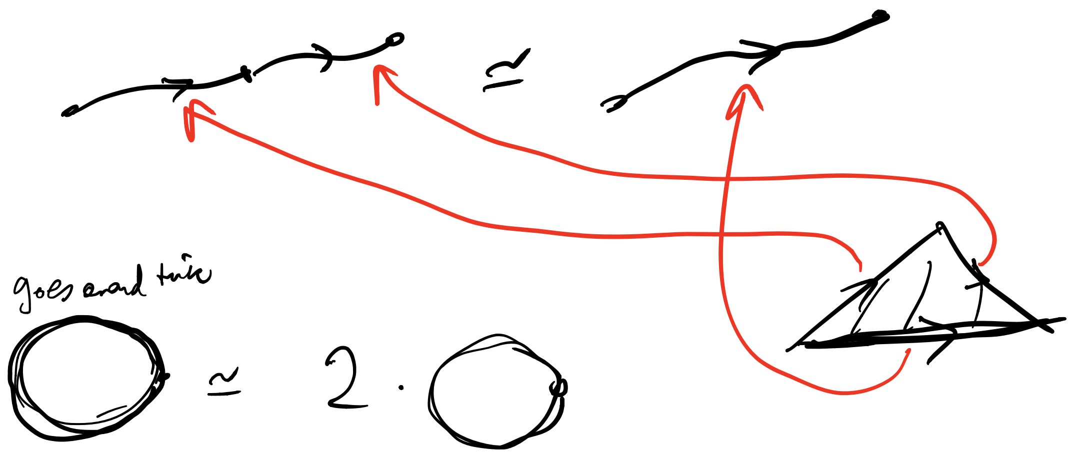
$\qquad 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$

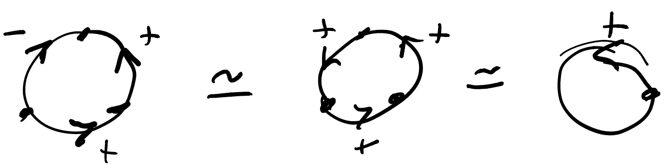
Induction: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n \end{cases}$

Ex. $\mathbb{RP}^2 =$  $\cup_{S^1} D^2$, $M \simeq S^1$

$$0 \to \underset{\substack{\| \text{ follows} \\ 0}}{H_2(\mathbb{RP}^2)} \to \underset{\substack{\| \\ \mathbb{Z}}}{H_1(S^1)} \xrightarrow[\text{key}]{\times 2} \underset{\substack{\| \\ \mathbb{Z}}}{H_1(M)} \oplus \underset{\substack{\| \\ 0}}{H_1(D^2)} \to \underset{\substack{\| \text{ follows} \\ \mathbb{Z}/2\mathbb{Z}}}{H_1(\mathbb{RP}^2)} \to \underset{\substack{\| \\ 0}}{\tilde{H}_0(S^1)}$$

$S^1 \to S^1$, $z \mapsto z^2$

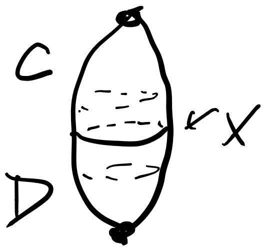


$H_1(S^1)$ generated by 

So $f: S^1 \to S^1$, $z \longmapsto z^2$, is mult. by 2 on $H_1(S^1)$.

Ex 1 of M-V)

$$SX = X \times [0,1] \Big/ \begin{matrix} X \times 0 \sim pt \\ X \times 1 \sim pt \end{matrix}$$



SX

C, D cones on X

$$SX = C \underset{X}{\cup} D$$

$$0 \to \tilde{H}_{i+1}(SX) \xrightarrow{\cong} \tilde{H}_i(X) \to 0$$

Naturality $\quad f : X \to Y \rightsquigarrow Sf : SX \to SY$

$$\begin{tikzcd}
\tilde{H}_{i+1}(SX) \arrow[r, "Sf_*"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_{i+1}(SY) \arrow[d, "\cong"] \\
\tilde{H}_i(X) \arrow[r, "f_*"] & \tilde{H}_i(Y)
\end{tikzcd}$$