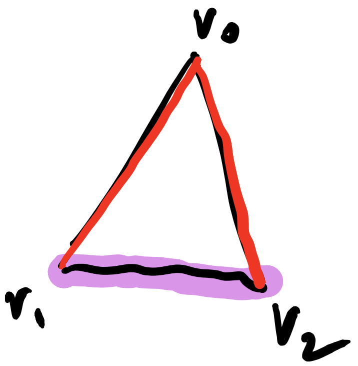
① Homology of $(B^n, \partial B^n)$

<u>Claim</u> $H_i(\Delta^n, \partial\Delta^n) = \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases}$

$[\mathbb{1} : \Delta^n \to \Delta^n]$ generates H_n.

Induction on n. $n = 0$ ✓



$\Lambda \subset \partial\Delta^n$, union of all faces except "bottom".

$(\Delta^n, \partial\Delta^n, \Lambda)$ LES homology

$$\begin{tikzcd}
H_i(\Delta^n,\Lambda) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_i(\Delta^n,\partial\Delta^n) \arrow[r, "\cong"'] & H_{i-1}(\partial\Delta^n,\Lambda) \arrow[r] & H_{i-1}(\Delta^n, \\
0 & & & 0 \arrow[u, phantom, "\shortparallel"] \\
& & H_{i-1}(\Delta^{n-1},\partial\Delta^{n-1}) \arrow[uu, "exc"', "\cong"] &
\end{tikzcd}$$

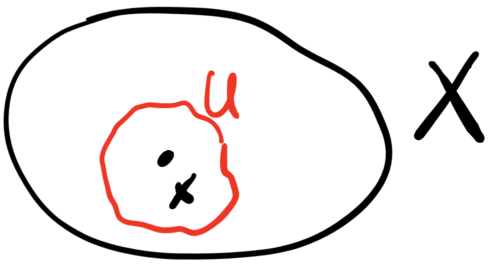
$X = A \cup B$ CW complexes

$$H_i(B, A \cap B) \xrightarrow[exc]{\cong} H_i(X, A)$$

② Local homology groups

$X \ni x$, $\{x\}$ closed.

$H_n(X, X-\{x\})$ local homology groups at x.



$$H_n(U, U-\{x\}) \xrightarrow[\text{exc}]{\cong} H_n(X, X-\{x\})$$

If (X,x) is homeomorphic to (Y,y)

$\Rightarrow H_n(X, X-x) \cong H_n(Y, Y-y)$

If x has a cone nbhd cL

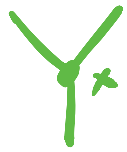
then $H_n(cL, cL-\{x\}) \cong \tilde{H}_{n-1}(cL-\{x\})$

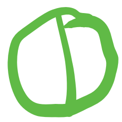
$= \tilde{H}_{n-1}(L)$

Ex: $\mathbb{R}^n$ every x has a cone nbhd with $L = S^{n-1}$

$$H_i(\mathbb{R}^n, \mathbb{R}^n - x) \cong \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases}$$

Cor. $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open, $\neq \phi$, homeomorphic
$\Rightarrow n = m$.

Ex.  $H_1(Y, Y - \{x\}) \cong \tilde{H}_0(3\text{ pts}) = \mathbb{Z} \oplus \mathbb{Z}$

Cor.   not homeomorphic.

③ If (X_α, x_α) NDR, connected.

Then $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\cong} \tilde{H}_n(\vee X_\alpha)$

Pf.

$$H_n\left(\coprod X_\alpha, \coprod \{x_\alpha\}\right) \underset{exc}{\cong} \tilde{H}_n(\vee X_\alpha)$$

$$\begin{gathered} \cong \\ \bigoplus H_n(X_\alpha, x_\alpha) \\ \cong \\ \bigoplus \tilde{H}_n(X_\alpha) \end{gathered}$$

④ Naturality

$$f:(X,A)\to(Y,B)$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & {}
\end{tikzcd}$$

all squares commute.

⑤ Attaching a cell.

$$Y = X \cup_f e^n.$$

(Y,X) is NDR

$$H_i(Y,X) \underset{\text{exc}}{\cong} \tilde{H}_i(Y/X) = \tilde{H}_i(S^n) = \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases}$$

$$H_{i+1}(Y,X) \to H_i(X) \to H_i(Y) \to H_i(Y,X)$$

If $i \neq n, n-1$ $\quad H_i(X) \xrightarrow{\cong} H_i(Y)$

The image of $H_n(Y,X) \to H_{n-1}(X)$
is generated by $f_*(z)$, $z \in \tilde{H}_{n-1}(S^{n-1})$

<u>Case 1</u> $f_*(z) = 0$

$$0 \to H_n(X) \to H_n(Y) \xrightarrow{\text{onto}} H_n(Y,X) \xrightarrow{0} H_{n-1}(X) \xrightarrow{\cong} H_{n-1}(Y) \to 0$$

$$H_n(Y,X) = \mathbb{Z}$$

$0 \to A \to B \to \mathbb{Z} \to 0$ splits

$$B = A \oplus \mathbb{Z}$$

$$H_n(Y) \cong H_n(X) \oplus \mathbb{Z}, \quad H_{n-1}(Y) \cong H_{n-1}(X)$$

<u>Case 2</u> $f_*(z)$ has order $k < \infty$.

$$H_n(Y) \cong H_n(X) \oplus \mathbb{Z}$$

$$H_{n-1}(Y) \cong H_n(X) / (\mathbb{Z}/k\mathbb{Z})$$

Case 3 $f_*(2)$ has infinite order.

$$0 \to H_n(X) \xrightarrow{\cong} H_n(Y) \xrightarrow{0} H_n(Y,X) \xrightarrow{1-1} H_{n-1}(X) \to H_{n-1}(Y) \to 0$$

$$H_{n-1}(Y) \cong H_{n-1}(X)/\mathbb{Z}$$

Cor. $\mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

⑥ Euler characteristic

X finite CW complex

$$\chi(X) = \sum_{n=0}^{\dim X} (-1)^n \cdot \#(\text{cells of dim } n)$$

$S^n = e^0 \cup e^n$ $\chi(S^n) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

$\dim X$

Thm $\chi(X) = \sum_{i\geq 0}^{\dim X} (-1)^i \operatorname{rank} H_i(X)$

Pf: Induction on # cells.

When adding an n-cell, RHS increases by $(-1)^n$.