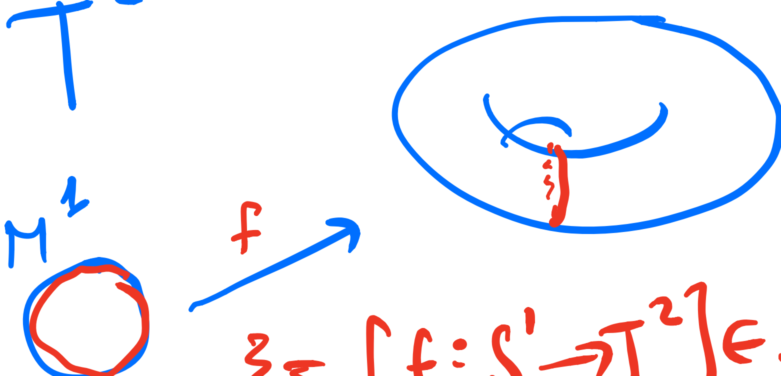
More on bordism

$$\Omega_n(X) = \{ f : M^n \to X \mid M \text{ closed smooth oriented} \} / \sim$$

$$\partial W = M_1 \sqcup (-M_2)$$

Examples

① $X = T^2$



$$\xi = [f : S^1 \to T^2] \in \Omega_1(T^2)$$

Claim $\xi \neq 0$.

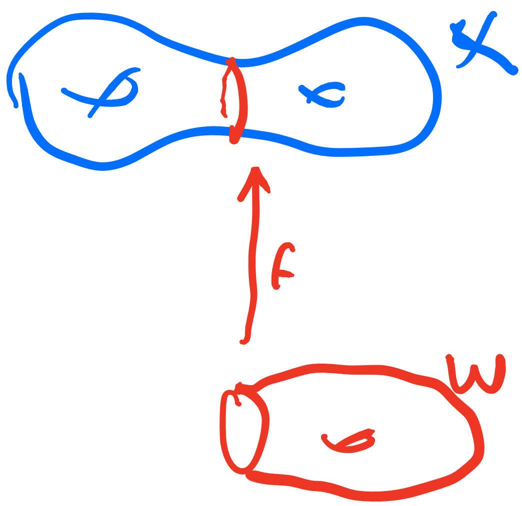
If $\xi = 0$ then $\exists\, W^2$

$F^{-1}(O) = $ 1-mfld in W, properly embedded $\pitchfork \partial W$

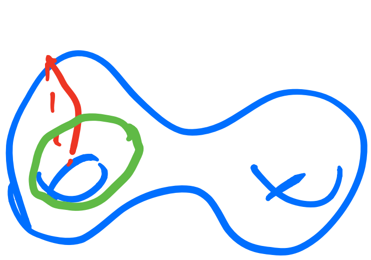
so $\partial F^{-1}(O)$ has an even # of pts

but $|\partial F^{-1}(O)| = |f^{-1}(O)| = 1$ $\times$.

Ex:

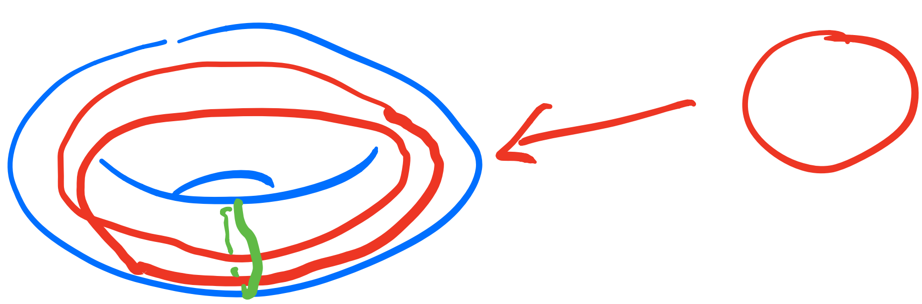


$[f] = 0 \in \mathfrak{N}_1(X)$



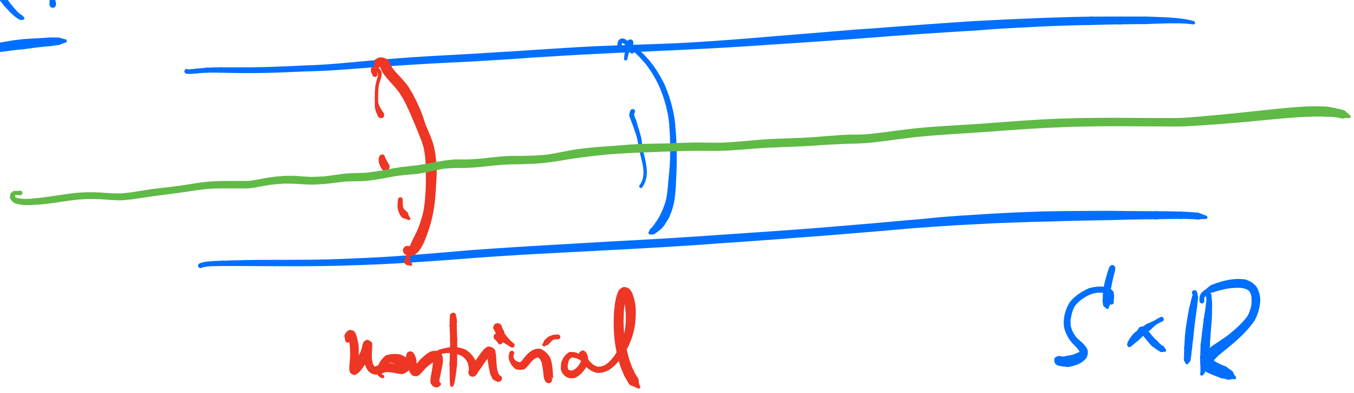
nontrivial!

Ex.

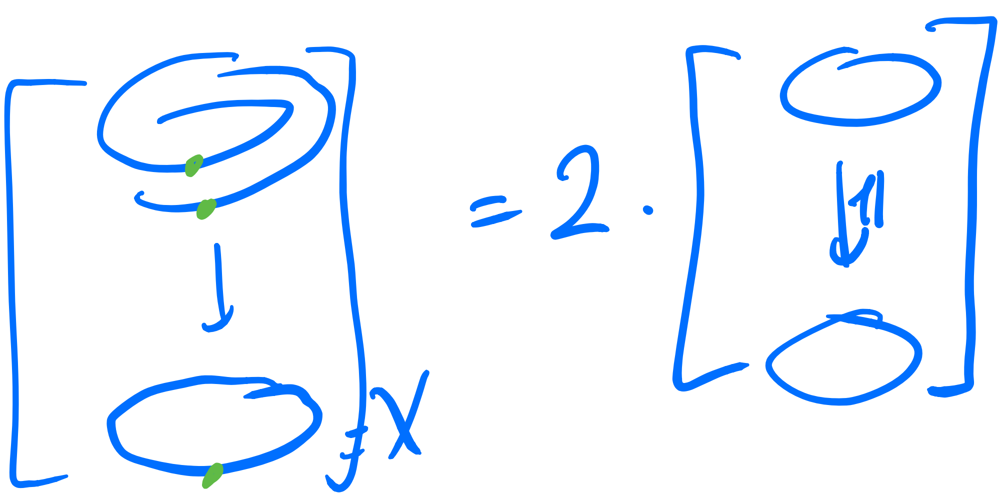


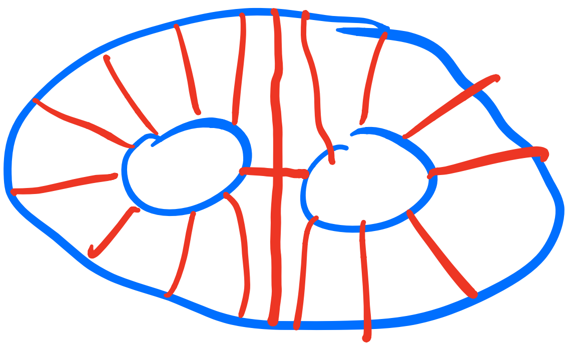
Everything is oriented. If Z is a compact oriented 1-mfld then $\Sigma \partial Z = 0$. But here get 2!

Ex.



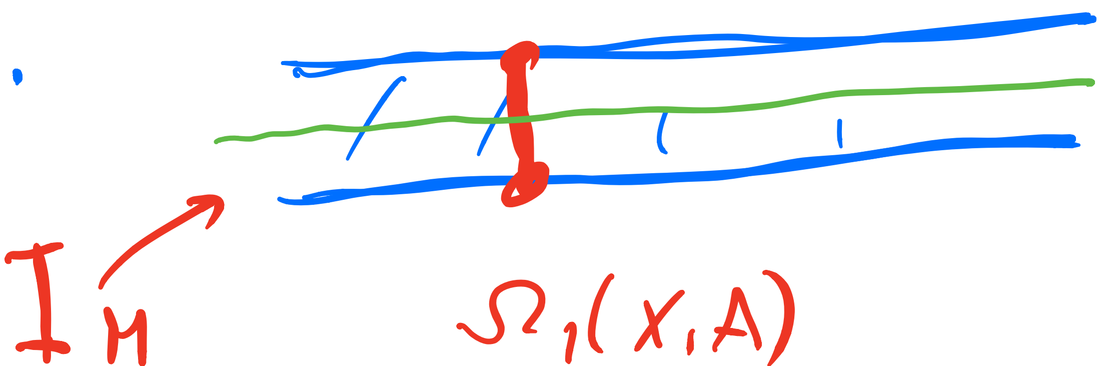
Ex.





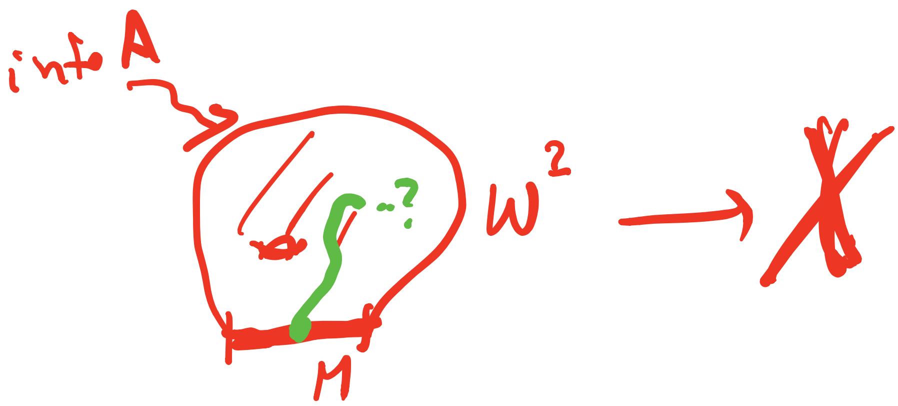
$$\Omega_n(X,A) = \left\{ (M^n, \partial M) \to (X,A) \;\middle|\; M \text{ compact, oriented} \right\} / \sim$$

Ex:



$X = \mathbb{R} \times I$

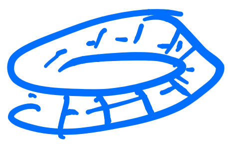
$A = \mathbb{R} \times \partial I$



Nontrivial class!

$\underline{\text{Ex.}}$ $[S^2 \xrightarrow{f} \mathbb{RP}^2] \in \Omega_2(\mathbb{RP}^2)$

$W = \mathrm{Map}(f) = S^2 \times I \big/ {(x,1)\sim f(x)} = \underline{\underset{\text{orientable}}{\mathbb{RP}^3 - (3\text{-ball})}}$!

For $S^1 \xrightarrow[\times 2]{} S^1$  not orientable

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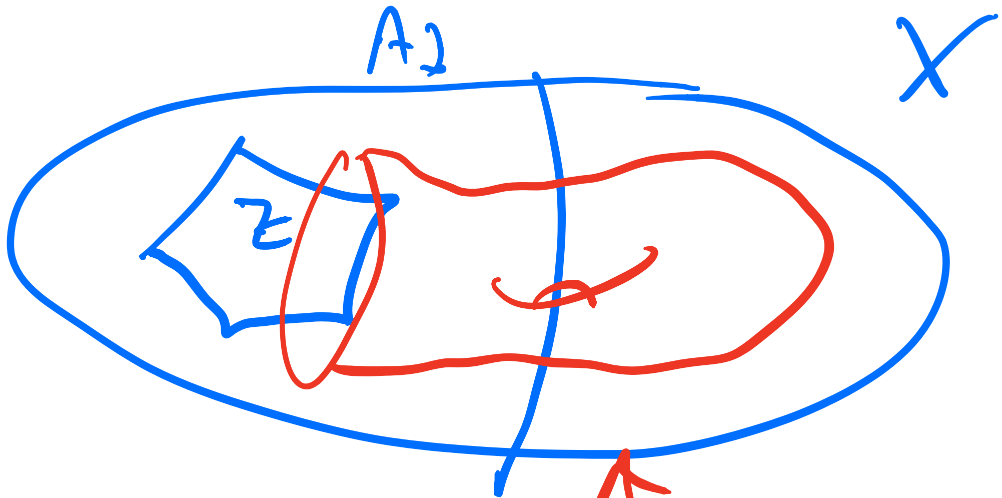
$\tilde{M} \xrightarrow{f} M$ double cover, $\tilde{M}, M$ connected,

$\tilde{M}$ orientable, M non-orientable

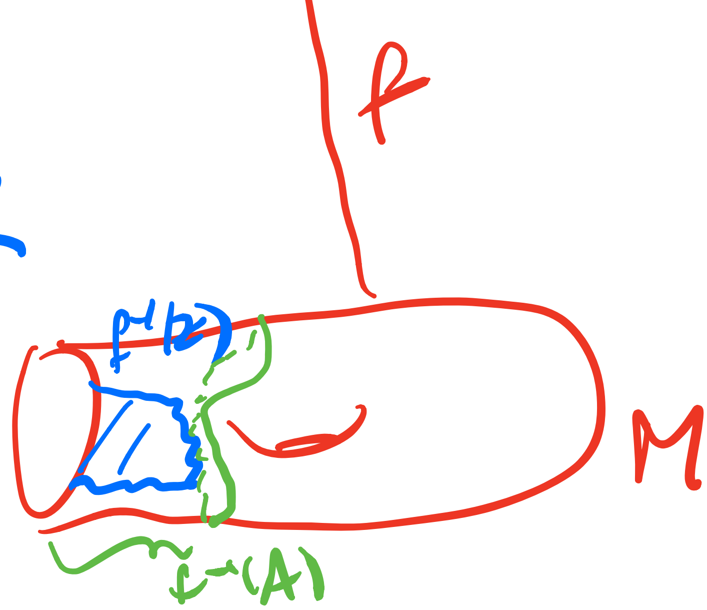
$\Rightarrow \mathrm{Map}(f)$ is orientable.

Excision for bordism



$\bar{Z} \subset \operatorname{int} A$.

$$\Omega_n(X-Z, A-Z) \xrightarrow{\cong} \Omega_n(X,A).$$



Sketch Onto

Let $\varphi : M \to [0,1]$ smooth,

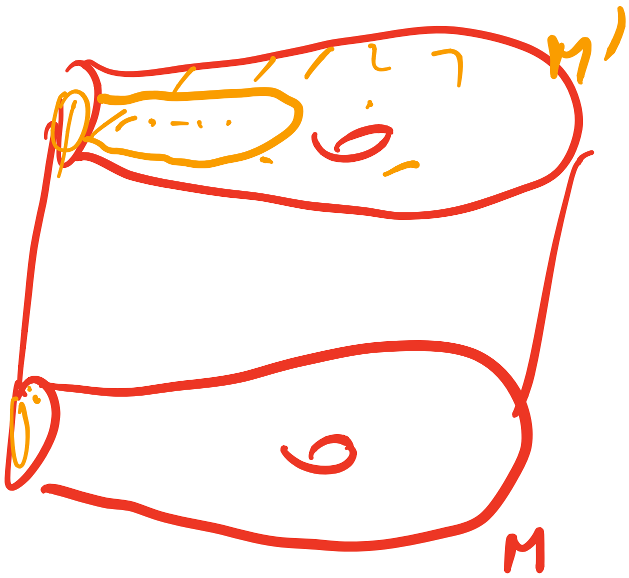
$\varphi = 0$ on $f^{-1}(Z)$, 1 outside $f^{-1}(A)$

(smooth Urysohn function).

$a \in (0,1)$ reg. value for φ and $\varphi/\partial M$.

M'

$f|(M', \partial M') \to (\cancel{X}-Z, A-Z)$



$M \times I = W$