Cell Complexes

1. For $n = 1, 2, \cdots$ let $f_n : S^1 \to S^1$ be the map $z \mapsto z^n$. Define

$$X(n) = S^1 \cup_{f_n} e^2$$

as the cell complex obtained by attaching a 2-cell to the circle via the attaching map f_n (this is an example of a *Moore space*). Similarly, for $m, n = 1, 2, \cdots$ define

$$X(m, n) = S^1 \cup_{f_m} e_1^2 \cup_{f_n} e_2^2$$

as the cell complex obtained from the circle by attaching two 2-cells, with attaching maps f_m and f_n.

(a) Show that if $m > n$ then the maps

$$f_m, f_{m-n} : S^1 \to S^1 \hookrightarrow X(n)$$

are homotopic as maps into $X(n)$.

(b) Show that for all m, n the space $X(m, n)$ is homotopy equivalent to the wedge sum $S^2 \vee X(k)$ for a certain $k = k(m, n)$. Give an explicit formula for k in terms of m, n.

2. Let X be a cell complex, A, B two of its subcomplexes such that $X = A \cup B$. Suppose $A, B, A \cap B$ are all contractible. Prove that X is contractible. N.B.: This is false for general spaces. E.g. the comb space

$$[0, 1] \times \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \cdots 0\} \times [0, 1]$$

is contractible, but the union of two copies along $(0, 1)$ is not.

3. Let X be a cell complex and

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

a sequence of subcomplexes such that $X = \cup_i X_i$. Suppose that each X_i is a retract of X_{i+1}. Prove that X_0 is a retract of X. Make sure you give an argument that your retraction is continuous.

4. Let X be the space obtained from the torus $S^1 \times S^1$ by identifying three distinct points to one point. Find an explicit cell structure on X.

5. In the following exercise we will show that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

   (a) Using linear algebra show that every element of $SO(3)$ is a rotation about an axis in $\mathbb{R}^3$.

   (b) For $x \in S^2$ and $\alpha \in [0, 2\pi]$ let $\phi_{x,\alpha}$ be the rotation about the axis $\mathbb{R}x$ by the angle α, where angles are measured using orientation of S^2.

   (c) Observe that $\phi_{x,0} = \phi_{x,2\pi} = \mathbb{1}$ so that ϕ defines a map

   $$S^2 \times [0, 2\pi]/\sim \to SO(3)$$

   where $S^2 \times \{0\}$ and $S^2 \times \{2\pi\}$ are collapsed to points. This quotient space can be identified with S^3.

   (d) Show that this map $S^3 \to SO(3)$ identifies antipodal points and induces a homeomorphism $\mathbb{R}P^3 \to SO(3)$.

6. Viewing the torus T as usual as the square $[0, 1]^2$ with opposite sides identified, let X be obtained from T by removing the open disk centered at $(\frac{1}{2}, \frac{1}{2})$ and with radius $\frac{1}{4}$. Find an explicit cell structure on X.

7. Show that $\mathbb{R}^2$ with n points removed is homotopy equivalent to the wedge sum of n circles.