Fundamental Group

1. Prove that a retract of a contractible space is contractible.

2. Prove that the torus minus one point deformation retracts to the wedge sum of two circles. Look up the definition of "wedge sum" in Hatcher.

3. Prove that $S^m * S^n = S^{m+n+1}$. Here $X * Y$ is the join of two spaces (look up the definition in Hatcher). Hint 1: Think about $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}$ and the unit spheres in these spaces. Hint 2: Prove that join is associative and then induct on $m+n$. For associativity, identify both $(X * Y) * Z$ and $X * (Y * Z)$ with a certain quotient of $X \times Y \times Z \times \Delta$ where Δ is a triangle.

4. Show that the following statements are equivalent for a space X:

   (a) Every map $S^1 \to X$ is null-homotopic (i.e. homotopic to a constant map).

   (b) Every map $S^1 \to X$ extends to $D^2 \to X$.

   (c) $\pi_1(X, x_0) = 1$ for every $x_0 \in X$.

5. A *topological group* is a set G which is both a topological space and a group, and the two structures are compatible in the sense that the group operations

   $$G \times G \to G, \qquad (x, y) \mapsto xy$$

   and

   $$G \to G, \qquad x \mapsto x^{-1}$$

   are continuous. For example, Lie groups are topological groups. If G is a topological group then the set of path-components $\pi_0(G)$ is also a group (while for a general topological space it is just a set).

   Prove that if G is a topological group then $\pi_1(G, 1)$ is abelian. Hint: For given $[f], [g] \in G$ construct a map $I \times I \to G$ of the square that agrees with f on the top and the bottom side, and agrees with g on the left and the right side.

   For example, $\pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$, $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. Also, $\pi_1(U(n)) = \mathbb{Z}$ while $\pi_1(SU(n)) = 1$ for all n. We may see this later in the class.

6. Prove that the following statements are equivalent for a path-connected space X:

(a) $\pi_1(X)$ is abelian. (Suppressing the basepoint here, they are all isomorphic.)

(b) for any two paths h, h' with the same endpoints we have $\beta_h = \beta_{h'}$. Thus $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ can be *canonically* identified for any two basepoints.

In general, β_h and $\beta_{h'}$ differ by conjugation.

7. Let X be a path-connected space with basepoint x_0. Consider the function

$$\Phi : \pi_1(X, x_0) \to [S^1, X]$$

to the set of homotopy classes of maps $S^1 \to X$ given by sending $[f]$ to the class of $S^1 = [0, 1]/0 \sim 1 \to X$ induced by f. Show that

(a) this is well-defined, i.e. does not depend on the choice of a representative f,

(b) Φ is onto, and

(c) $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Thus $[S^1, X]$ can be thought of as the set of conjugacy classes in $\pi_1(X, x_0)$. Note that that's not necessarily a group.