Definitions and examples of manifolds

Guillemin-Pollack

In this part we use the Guillemin-Pollack definitions, in particular of a smooth function defined on a subset $X \subset \mathbb{R}^n$: $f : X \to \mathbb{R}^m$ is smooth if for every $x \in X$ there is a neighborhood $U$ of $x$ in $\mathbb{R}^n$ and a smooth function $F_U : U \to \mathbb{R}^m$ that agrees with $f$ on $U \cap X$.

1. For $k < n$ view $\mathbb{R}^k$ as a subset of $\mathbb{R}^n$ via $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$. Show that a function $f : \mathbb{R}^k \to \mathbb{R}$ is smooth (using the G-P subset definition) if and only if it is smooth in the usual sense.

2. Let $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^k, Z \subset \mathbb{R}^n$. If $f : X \to Y$ and $g : Y \to Z$ are smooth (as maps to the ambient Euclidean spaces) then so is the composition $gf : X \to Z$. If $f, g$ are diffeomorphisms so is $gf$.

3. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m, X' \subset \mathbb{R}^k, Y' \subset \mathbb{R}^l$. If $f : X \to X'$ and $g : Y \to Y'$ are smooth, so is $f \times g : X \times Y \to X' \times Y'$. Here we view $X \times Y$ as a subset of $\mathbb{R}^{n+m}$ and $X' \times Y'$ as a subset of $\mathbb{R}^{k+l}$.

4. Let $X \subset \mathbb{R}^n$ and let $\Delta = \{(x, x) \in \mathbb{R}^{2n} \mid x \in X\}$ be the diagonal. Show that $\Delta$ is diffeomorphic to $X$. More generally, if $f : X \to Y$ is smooth, then the graph of $f$

$$\text{graph}(f) = \{(x, f(x)) \mid x \in X\}$$

is diffeomorphic to $X$. I am suppressing various ambient Euclidean spaces.

5. Show that the map $a : S^n \to S^n$ defined by $a(x) = -x$ is a diffeomorphism.

6. Show that the letters $L$ and $I$ are homeomorphic but not diffeomorphic. Here $I = \{0\} \times \mathbb{R}$ and $L = \{0\} \times [0, \infty) \cup [0, \infty) \times \{0\}$. Hint: View $I$ as a subset of $\mathbb{R}$ and a diffeomorphism $I \to L$ as a local parametrization around the corner point, the derivative should be injective.

Chart definition

7. Show that $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is not a topological manifold. Hint: How many components does $X \setminus \{(0, 0)\}$ have?
8. Let $X$ be a smooth manifold. Show that the set $C^\infty(X)$ of all smooth functions $X \to \mathbb{R}$ is an algebra, i.e. if $f, g \in C^\infty(X)$ then so are $af + bg$ and $fg$ for any $a, b \in \mathbb{R}$. You can use the fact that this is so when $X$ is an open set in $\mathbb{R}^n$.

9. The group $GL_n(\mathbb{R})$ of $n \times n$ matrices is naturally an open set in the space $M(n)$ of all real $n \times n$ matrices, which is naturally identified with $\mathbb{R}^{n^2}$. Show that the maps $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ defined by $(A, B) \mapsto AB$ and $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ are smooth. This shows that $GL_n(\mathbb{R})$ is a Lie group.

The Regular Value Theorem, Lie groups

10. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = xyz + x^3 + y^3 + z^3$ and let $M_a = f^{-1}(a)$. Show that $M_a$ is a manifold for $a \neq 0$, and that $M_0 \setminus \{(0, 0, 0)\}$ is a manifold.

11. The unitary group $U(n)$ is the group of complex $n \times n$ matrices $M$ such that $MM^* = I$, where $M^*$ is transpose followed by conjugation of all entries. Show that $U(n)$ is a compact Lie group.

12. The (real) symplectic group $Sp(2n, \mathbb{R})$ is the group of real $2n \times 2n$ matrices $M$ that satisfy $M\Omega M^T = \Omega$, where $\Omega$ is the block matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with blocks of size $n \times n$ and $I_n$ the identity matrix. In other words, if you think of $\Omega$ as defining a skew-symmetric bilinear form $(v, w) \mapsto v^t\Omega w$ on $\mathbb{R}^{2n}$, then the condition is that $M$ preserves this form. Show that $Sp(2n, \mathbb{R})$ is a Lie group. Also show that $Sp(2, \mathbb{R}) = SL_2(\mathbb{R})$ (the form $(v, w) \mapsto v^t\Omega w$ is the signed area of the parallelogram spanned by $v$ and $w$). Hint: Use the argument for $O(n)$ as a template. This time the range of the function $F$ is the space of skew-symmetric matrices $S$, those with $S^t = -S$. 