Homework 5: Singularities, Residue Calculus

Singularities

1. If $f$ is a holomorphic function defined in $\{|z| > R\}$ (we think of this set as a neighborhood of $\infty$) we say that $\infty$ is a removable or essential singularity or a pole provided that $0$ is the respective singularity for the function $g(z) = f(\frac{1}{z})$. Show that

   (i) A nonconstant polynomial has a pole at infinity.

   (ii) If $f$ is an entire function which is not a polynomial, then $f$ has an essential singularity at $\infty$.

Note that our standard essential singularities such as $e^{\frac{1}{z}}$ or $\sin \frac{1}{z}$ come from the construction in (ii), and we also get some new examples, e.g. $\frac{e^{\frac{1}{z}}}{\sin \frac{1}{z}}$ etc.

Residue Calculus.

2. Let $P$ be a polynomial of degree $\geq 2$.

   (a) Show that for any circle $C$ of big enough radius so that it encloses all roots we have

   \[ \int_C \frac{dz}{P(z)} = 0 \]

   (b) Assuming all roots $z_1, \ldots, z_n$ of $P$ are distinct prove (using the Residue theorem) that

   \[ \sum_{j=1}^{n} \frac{1}{P'(z_j)} = 0 \]

   You may want to spend a few minutes thinking about how to prove (ii) without the Residue theorem.

3. Compute

   \[ \frac{i}{4} \int_{|z|=2023} \tan(\pi z) dz \]

4. Compute

   \[ \frac{1}{2\pi i} \int_{|z|=1} \sin \left( \frac{1}{z} \right) dz \]
5. Show that
\[ \int_0^{2\pi} \frac{d\theta}{1 - 2r \cos \theta + r^2} = \frac{2\pi}{|1 - r^2|} \]
when \( r \in \mathbb{R} \setminus \{-1, 1\} \).

6. Prove the Wallis formula
\[ \frac{1}{2\pi} \int_0^{2\pi} (2 \cos \theta)^{2m} d\theta = \left(\frac{2m}{m}\right) \]
for \( m = 1, 2, \cdots \).

7. Prove that
\[ \int_0^{\infty} dx \frac{1}{1 + x^n} = \frac{\pi}{n \sin \left(\frac{\pi}{n}\right)} \]
for \( n = 2, 3, \cdots \).

Hint: Use the contour consisting of \([0, R], [0, Re^{2\pi i/n}]\) and the short arc of \(|z| = R\) connecting the endpoints.

8. Prove that
\[ \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + 1)^2} dx = \frac{\pi}{2e} \]

Hint: As usual, \( f(z) = \frac{z^3 e^{iz}}{(z^2 + 1)^2} \).

9. Prove that
\[ \int_0^{\infty} \frac{x \sin x}{x^4 + 1} dx = \frac{\pi}{2} e^{-1/\sqrt{2}} \sin \left(\frac{1}{\sqrt{2}}\right) \]

10. Prove that
\[ \int_{-\infty}^{\infty} \frac{\cos x}{\cosh x} dx = \frac{\pi}{\cosh \left(\frac{\pi}{2}\right)} \]

Hint: For the contour take the rectangle of height \( \pi \) and base \([-R, R]\).

11. Prove that
\[ \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi^4}{90} \]

This can be done in many ways, including Fourier series, but here you should use Residue Calculus.

12. Prove that
\[ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{12} \]

Hint: \( \pi \csc \pi z = \frac{\pi}{\sin \pi z} \) has residue \((-1)^n\) at \( z = n \).