Banach Spaces

1. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that if $1 \leqslant q < p \leqslant \infty$ then $L^q(X) \supseteq L^p(X)$. Moreover, if X contains an infinite family of pairwise disjoint subsets of positive measure, then $L^q(X) \supsetneq L^p(X)$.

2. Show that for little ℓ's the reverse inclusion holds: if $1 \leqslant q < p \leqslant \infty$ then $\ell^q \subsetneq \ell^p$.

3. In Problem 1. show that inclusion $I : L^p(X) \to L^q(X)$ is a bounded linear operator and determine its operator norm.

4. In Problem 2. show that inclusion $J : \ell^q \to \ell^p$ is a bounded linear operator and determine its operator norm.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leqslant p, q, r < \infty$, $f \in L^p(X)$, $g \in L^q(X)$. Show that if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ then $\|fg\|_r \leqslant \|f\|_p \|g\|_q$.

In the problems below the measure is always Lebesgue unless specified otherwise.

6. Let $1 \leqslant p, q, r < \infty$ and assume there is a constant $c > 0$ such that $\|fg\|_r \leqslant c\|f\|_p\|g\|_q$ for all $f \in L^p(\mathbb{R}^n)$ and all $g \in L^q(\mathbb{R}^n)$. Show that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

7. True or false: Suppose $1 < r < \infty$ and $f \in L^p([0,1])$ for all $1 \leqslant p < r$. Then $f \in L^r([0,1])$ and $\|f\|_r = \lim_{p \to r^-} \|f\|_p$.

8. True or false: $L^\infty([0,1]) = \bigcap_{p \in [1,\infty)} L^p([0,1])$.

9. Show that $L^p(\mathbb{R}) \nsubseteq L^q(\mathbb{R})$ for $p \neq q$.

10. Let $1 \leqslant p < \infty$. The goal of this exercise is to show that $L^p(\mathbb{R})$ and $L^p([0,1])$ are isometric Banach spaces (i.e. there is an isomorphism that preserves the norm). Both $\mathbb{R}$ and $[0,1]$ are equipped with Lebesgue measure m.

    (a) Choose a continuous function $f : \mathbb{R} \to (0, \infty)$ such that $\int_{\mathbb{R}} f\, dm = 1$ and let μ be the probability measure on $\mathbb{R}$ defined by

    $$d\mu = f\ dm$$

Construct an isometric isomorphism

$$L^p(\mathbb{R}) \to L^p(\mu)$$

of the form $g \mapsto gh$ for a suitable function h.

(b) Show that there is a homeomorphism $\phi : \mathbb{R} \to (0,1)$ such that $m(\phi(E)) = \mu(E)$ for every measurable E.

(c) Conclude.