Lebesgue Differentiation

Points of density

One way to prove that a set in $\mathbb{R}^n$ has Lebesgue measure 0 is to show that it does not have any points of density. (This is a hint for problems 2. and 3.)

1. Let $A \subset \mathbb{R}$ be a measurable set so that 0 is a point of density for $A$. Show that for every $\lambda \in \mathbb{R} \setminus \{0\}$ there is an infinite sequence $x_k \to 0$ with $x_k \neq 0$ and so that $x_k, \lambda x_k \in A$.

2. Let $K \subset \mathbb{R}^n$ be a compact set and $r > 0$. Show that the set
   \[ S_r = \{ x \in \mathbb{R}^n \mid d(x, K) = r \} \]
   has measure 0. Here the distance $d(x, K) = \min\{d(x, y) \mid y \in K\}$.

3. Let $A \subset \mathbb{R}^n$ have positive Lebesgue measure. Show that for any countable dense set $D \subset \mathbb{R}^n$ the complement of
   \[ \bigcup_{y \in D} (A + y) \]
   has measure 0.

4. Construct a measurable set $A \subset \mathbb{R}$ such that
   \begin{enumerate}
   \item \[ \lim_{r \to 0} \frac{m(A \cap (-r, r))}{2r} = d \]
   for a given $d \in (0, 1)$,
   \item the limit in (a) does not exist.
   \end{enumerate}

Ergodic group actions

Let $(X, \mathcal{M}, \mu)$ be a measure space and $G$ a group of measure preserving bijections of $X$. For example, $SL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ preserving the Lebesgue measure, and so does any subgroup. Such an action is \textit{ergodic} if the following holds: if $A \subset X$ is measurable and $G$-invariant, then $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. 

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5. Show that the action of $G$ on $X$ is ergodic if and only if every measurable function $f : X \to \mathbb{R}$ which is $G$-invariant (meaning that $f(gx) = f(x)$ for every $x \in X$ and $g \in G$) is necessarily constant a.e. Hint: Show that if a measurable function is not constant a.e. then there is some $c \in \mathbb{R}$ so that $f^{-1}(c, \infty)$ and $f^{-1}(-\infty, c)$ both have positive measure.

6. Prove that the action of $G = \mathbb{Q}$ by translation on $\mathbb{R}$ is ergodic.

Notes: The same is true (with the same proof) for any dense subgroup of the group of translations on $\mathbb{R}^n$, and for any dense subgroup of the group of rotations of the circle. In class I said that you need points of density here, and the weaker version we had before is not sufficient because you might get intervals of different sizes. I changed my mind, and I think you can use the older fact, but of course that works only in dimension 1.

7. Suppose $\mu$ is a Borel measure on $\mathbb{R}$ and $\mu \ll m$ where $m$ is Lebesgue measure. If $\mu$ is invariant under rational translations, show that it is a constant multiple of $m$. Hint: Radon-Nikodym plus ergodicity. This is also true without assuming $\mu \ll m$ but we can’t prove it at this point.

**Bounded Variation**

8. Suppose $f, g : [a, b] \to \mathbb{R}$ are functions of bounded variation. Show that the product $fg$ has bounded variation.

9. Suppose $f : [a, b] \to [c, \infty)$ has bounded variation and $c > 0$. Show that $\frac{1}{f}$ has bounded variation.

**Cantor sets**

10. We’ve seen in class that the Cantor function associated to the middle thirds Cantor set is not absolutely continuous, much less Lipschitz. Show that there is a Cantor set defined similarly as the middle thirds Cantor sets except for the sizes of removed intervals, so that the associated Cantor function is Lipschitz.