Lebesgue Differentiation

Points of density

One way to prove that a set in $\mathbb{R}^n$ has Lebesgue measure 0 is to show that it does not have any points of density. (This is a hint for problems 2. and 3.)

1. Let $A \subset \mathbb{R}$ be a measurable set so that 0 is a point of density for A. Show that for every $\lambda \in \mathbb{R} \setminus \{0\}$ there is an infinite sequence $x_k \to 0$ with $x_k \neq 0$ and so that $x_k, \lambda x_k \in A$.

2. Let $K \subset \mathbb{R}^n$ be a compact set and $r > 0$. Show that the set

$$S_r = \{x \in \mathbb{R}^n \mid d(x, K) = r\}$$

   has measure 0. Here the distance $d(x, K) = \min\{d(x, y) \mid y \in K\}$.

3. Let $A \subset \mathbb{R}^n$ have positive Lebesgue measure. Show that for any countable dense set $D \subset \mathbb{R}^n$ the complement of

$$\bigcup_{y \in D} (A + y)$$

   has measure 0.

4. Construct a measurable set $A \subset \mathbb{R}$ such that

   (a)
$$\lim_{r \to 0} \frac{m(A \cap (-r, r))}{2r} = d$$
   for a given $d \in (0, 1)$,

   (b) the limit in (a) does not exist.

Ergodic group actions

Let $(X, \mathcal{M}, \mu)$ be a measure space and G a group of measure preserving bijections of X. For example, $SL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ preserving the Lebesgue measure, and so does any subgroup. Such an action is *ergodic* if the following holds: if $A \subset X$ is measurable and G-invariant, then $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

5. Show that the action of G on X is ergodic if and only if every measurable function $f : X \to \mathbb{R}$ which is G-invariant (meaning that $f(gx) = f(x)$ for every $x \in X$ and $g \in G$) is necessarily constant a.e. Hint: Show that if a measurable function is not constant a.e. then there is some $c \in \mathbb{R}$ so that $f^{-1}(c, \infty)$ and $f^{-1}(-\infty, c)$ both have positive measure.

6. Prove that the action of $G = \mathbb{Q}$ by translation on $\mathbb{R}$ is ergodic.

   Notes: The same is true (with the same proof) for any dense subgroup of the group of translations on $\mathbb{R}^n$, and for any dense subgroup of the group of rotations of the circle. In class I said that you need points of density here, and the weaker version we had before is not sufficient because you might get intervals of different sizes. I changed my mind, and I think you can use the older fact, but of course that works only in dimension 1.

7. Suppose μ is a Borel measure on $\mathbb{R}$ and $\mu \ll m$ where m is Lebesgue measure. If μ is invariant under rational translations, show that it is a constant multiple of m. Hint: Radon-Nikodym plus ergodicity. This is also true without assuming $\mu \ll m$ but we can't prove it at this point.

Bounded Variation

8. Suppose $f, g : [a, b] \to \mathbb{R}$ are functions of bounded variation. Show that the product fg has bounded variation.

9. Suppose $f : [a, b] \to [c, \infty)$ has bounded variation and $c > 0$. Show that $\frac{1}{f}$ has bounded variation.

Cantor sets

10. We've seen in class that the Cantor function associated to the middle thirds Cantor set is not absolutely continuous, much less Lipschitz. Show that there is a Cantor set defined similarly as the middle thirds Cantor sets except for the sizes of removed intervals, so that the associated Cantor function is Lipschitz.