Signed measures and Lebesgue-Radon-Nikodym

Signed measures and Hahn-Jordan

1. Let ν_1, ν_2 be two finite signed measures on a space X. Show that $\|\nu_1 + \nu_2\| \leqslant \|\nu_1\| + \|\nu_2\|$ and that $\|-\nu_1\| = \|\nu_1\|$, where $\|\cdot\|$ denotes total variation (so $\|\nu\| = \nu^+(X) + \nu^-(X)$ if $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν).

2. Show that if $\nu = \mu - \rho$ where μ, ρ are positive measures (of which one is finite), then $\mu \geqslant \nu^+, \rho \geqslant \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of the signed measure ν.

Lebesgue-Radon-Nikodym

3. Let $f \in L^1([0,1])$ be an integrable function and define $F : [0,1] \to \mathbb{R}$ by
$$F(x) = \int_0^x f \, dx$$
Suppose $E \subset [0,1]$ is measurable. Show that $F(E)$ is also measurable and that $m(F(E)) \leqslant \int_E |f(x)| \, dx$.

   Hint: Recall that F is continuous (earlier homework). It's not true that the image of a measurable set by a continuous function is always measurable. First prove the inequality when E is an interval, and then when it is a countable union of intervals. Use that $E \to \int_E |f(x)| \, dx$ is a measure absolutely continuous with respect to Lebesgue.

4. If μ and ν are finite positive measures on the measurable space $(X, \mathcal{M})$, show that there is a nonnegative measurable function f on X such that for all E in $\mathcal{M}$
$$\int_E (1 - f) \, d\mu = \int_E f \, d\nu$$

5. Let m be Lebesgue measure on the real line $\mathbb{R}$, and for each Lebesgue measurable subset E of $\mathbb{R}$ define
$$\mu(E) = \int_E \frac{1}{1 + x^2} \, dm(x)$$
Show that m is absolutely continuous with respect to μ and compute the Radon-Nikodym derivative $dm/d\mu$.

6. Let μ and ν be finite positive measures on the measurable space $(X, \mathcal{M})$ such that $\nu \ll \mu \ll \nu$, and let $f = d\nu/d(\mu+\nu)$ represent the Radon-Nikodym derivative of ν with respect to $\mu+\nu$. Show that $0 < f(x) < 1$ for a.e. x with respect to μ.

7. Let μ and ν be two positive measures on a measurable space $(X, \mathcal{M})$. Suppose for every $\epsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and $\nu(E^c) < \epsilon$. Show that $\mu \perp \nu$.

8. Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{L}, m)$. Let ν be the counting measure on $\mathcal{L}$, that is, ν is defined by setting $\nu(E) \in [0, \infty]$ to be equal to the cardinality of E

   (a) Show that $m \ll \nu$ but $dm/d\nu$ does not exist.

   (b) Show that ν does not have a Lebesgue decomposition with respect to m.

9. For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$