Signed measures and Lebesgue-Radon-Nikodym

Signed measures and Hahn-Jordan

1. Let \( \nu_1, \nu_2 \) be two finite signed measures on a space \( X \). Show that \( \| \nu_1 + \nu_2 \| \leq \| \nu_1 \| + \| \nu_2 \| \) and that \( \| - \nu_1 \| = \| \nu_1 \| \), where \( \| \cdot \| \) denotes total variation (so \( \| \nu \| = \nu^+(X) + \nu^-(X) \) if \( \nu = \nu^+ - \nu^- \) is the Jordan decomposition of \( \nu \)).

2. Show that if \( \nu = \mu - \rho \) where \( \mu, \rho \) are positive measures (of which one is finite), then \( \mu \geq \nu^+, \rho \geq \nu^- \), where \( \nu = \nu^+ - \nu^- \) is the Jordan decomposition of the signed measure \( \nu \).

Lebesgue-Radon-Nikodym

3. Let \( f \in L^1([0,1]) \) be an integrable function and define \( F : [0,1] \to \mathbb{R} \) by

\[
F(x) = \int_0^x f \, dx
\]

Suppose \( E \subset [0,1] \) is measurable. Show that \( F(E) \) is also measurable and that \( m(F(E)) \leq \int_E |f(x)| \, dx \).

Hint: Recall that \( F \) is continuous (earlier homework). It’s not true that the image of a measurable set by a continuous function is always measurable. First prove the inequality when \( E \) is an interval, and then when it is a countable union of intervals. Use that \( E \to \int_E |f(x)| \, dx \) is a measure absolutely continuous with respect to Lebesgue.

4. If \( \mu \) and \( \nu \) are finite positive measures on the measurable space \( (X, \mathcal{M}) \), show that there is a nonnegative measurable function \( f \) on \( X \) such that for all \( E \) in \( \mathcal{M} \)

\[
\int_E (1 - f) \, d\mu = \int_E f \, d\nu
\]

5. Let \( m \) be Lebesgue measure on the real line \( \mathbb{R} \), and for each Lebesgue measurable subset \( E \) of \( \mathbb{R} \) define

\[
\mu(E) = \int_E \frac{1}{1 + x^2} \, dm(x)
\]

Show that \( m \) is absolutely continuous with respect to \( \mu \) and compute the Radon-Nikodym derivative \( dm/d\mu \).
6. Let $\mu$ and $\nu$ be finite positive measures on the measurable space $(X, \mathcal{M})$ such that $\nu \ll \mu \ll \nu$, and let $f = dv/d(\mu + \nu)$ represent the Radon-Nikodym derivative of $\nu$ with respect to $\mu + \nu$. Show that $0 < f(x) < 1$ for a.e. $x$ with respect to $\mu$.

7. Let $\mu$ and $\nu$ be two positive measures on a measurable space $(X, \mathcal{M})$. Suppose for every $\epsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and $\nu(E^c) < \epsilon$. Show that $\mu \perp \nu$.

8. Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{L}, m)$. Let $\nu$ be the counting measure on $\mathcal{L}$, that is, $\nu$ is defined by setting $\nu(E) \in [0, \infty]$ to be equal to the cardinality of $E$

   (a) Show that $m \ll \nu$ but $dm/d\nu$ does not exist.

   (b) Show that $\nu$ does not have a Lebesgue decomposition with respect to $m$.

9. For $j = 1, 2$, let $\mu_j, \nu_j$ be $\sigma$-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{dv_1}{d\mu_1}(x_1) \frac{dv_2}{d\mu_2}(x_2)$$