Modes of convergence, Fubini

Modes of convergence

1. Prove the following variant of Egorov's theorem that applies to sets of infinite measure. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $g \in L^1(\mu)$. Suppose $f_n : X \to \mathbb{R}$ are measurable with $|f_n| \leqslant g$ and $f_n \to f$ pointwise. Then for every $\epsilon > 0$ there is a set $E \subset X$ of measure $< \epsilon$ so that $f_n \to f$ uniformly on $X \smallsetminus E$.

   Hint: We are not assuming here that X is σ-finite, but g will be 0 outside a σ-finite subset, and in fact it will be $< \delta$ outside a set of finite measure.

2. Recall the Borel-Cantelli lemma: if $\sum_i \mu(A_i) < \infty$ then the set of points that belong to infinitely many A_i has measure 0. Find an example of a collection of measurable sets $A_i \subset [0, 1]$ so that $\mu(A_i) \to 0$ and so that every point in $[0, 1]$ belongs to infinitely many A_i.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n, f : X \to [0, \infty)$ integrable functions (recall that this means the integrals are finite). Assume that $f_n \to f$ pointwise as $n \to \infty$. Show that

$$\int_X f_n \, d\mu - ||f_n - f|| \to \int_X f \, d\mu$$

   where $|| \cdot ||$ denotes the L^1-norm.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let f_i, f be measurable functions. Prove that $f_i \to f$ in measure iff each subsequence of f_i has a further subsequence that converges to f a.e.

5. A sequence f_i of real-valued functions on a measure space $(X, \mathcal{M}, \mu)$ is *Cauchy in measure* if for every $\epsilon > 0$

$$\mu(\{x \in X \mid |f_i(x) - f_j(x)| > \epsilon\}) \to 0$$

   as $i, j \to \infty$. Show that if f_i is Cauchy in measure then there is a measurable function $f : X \to \mathbb{R}$ so that $f_i \to f$ in measure.

   Hint: Find a sequence $i_1 < i_2 < \cdots$ so that $\mu(\{|f_{i_j} - f_{i_{j+1}}| > 2^{-j}\} < 2^{-j}$ and argue that $f_{i_j} \to f$ a.e.

Fubini-Tonelli (sometimes called Fubinelli)

6. Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{S})$ be measurable spaces. We endow $X \times Y$ with the σ-algebra $\mathcal{M} \otimes \mathcal{N}$. Show that

   (a) projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are measurable,

   (b) $F : Z \to X \times Y$ is measurable iff compositions $p_X F$ and $p_Y F$ are measurable.

7. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty)$. Let

$$G_f = \{(x, t) \in X \times \mathbb{R} \mid 0 \leqslant t \leqslant f(x)\}$$

   Prove that f is measurable if and only if G_f is measurable (with respect to the product measure and with Lebesgue measure m on Borel subsets of $\mathbb{R}$) and if so then

$$(\mu \times m)(G_f) = \int f \; d\mu$$

   Hint: Consider $X \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $(x, t) \mapsto (f(x), t)$.

8. (Distribution Theorem) In the setting of Problem 7 and assuming $f \in L^+$ prove that

$$\int f \; d\mu = \int_{[0,\infty)} \mu(\{x \in X \mid f(x) \geqslant t\}) dt$$

9. Consider the function

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

   defined on $[0, 1] \times [0, 1] \backslash \{(0, 0)\}$. Discuss the existence and equality of the three integrals

$$\int_{[0,1]\times[0,1]} f, \qquad \int_{[0,1]} \int_{[0,1]} f \; dx \; dy, \qquad \int_{[0,1]} \int_{[0,1]} f \; dy \; dx$$

   Hint: An antiderivative of the given function with respect to x is $-\frac{x}{x^2+y^2}$.

10. Consider $X = [0, 1]$ with Lebesgue measure m defined on the Borel σ-algebra $\mathcal{B}$, and also the counting measure ν defined on the set $\mathcal{P}$ of all subsets of X. Let $D = \{(x, x) \mid x \in X\}$ be the diagonal.

(a) Show that D is in the σ-algebra $\mathcal{B} \otimes \mathcal{P}$.

(b) Show that the two integrals in Baby Fubini are different.

(c) Explain. What is the measure of D?