More measurable functions and integration

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function (meaning that for every Borel set $E \subset \mathbb{R}$ the set $f^{-1}(E)$ is Lebesgue measurable; we always use the Borel σ-algebra in the codomain). Show that there is a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e. Hint: Simple, positive, general.

2. Let $A, B \subset [0,1]$ be measurable. Show that the function $\Phi : [0,1] \to [0,1]$ defined by
$$\Phi(t) = m((A + t) \cap B)$$
is continuous.

   Notes: This is an important feature of Lebesgue measure and it is used in proofs of fundamental theorems in ergodic theory, e.g. ergodicity of the horocycle flow.

   Hint for one proof: Regularity

   Hint for a different proof: Monotone class theorem.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose $A_1, A_2, \cdots, A_n \in \mathcal{M}$, i.e. they are measurable subsets of X.

   (a) If every $x \in X$ belongs to at most k of these sets, show that $\sum_{i=1}^{n} \mu(A_i) \leq k$.

   (b) If every $x \in X$ belongs to at least k of these sets, show that $\sum_{i=1}^{n} \mu(A_i) \geq k$.

   Hint: Reformulate the problem in terms of the indicator functions 1_{A_i}.

4. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Let g_n and G_n be the Darboux approximations with respect to a fixed sequence of finer and finer subdivisions, as in the class, so that
$$g_1 \leq g_2 \leq \cdots \leq f \leq \cdots \leq G_2 \leq G_1$$
and set $g = \lim_{n\to\infty} g_n$ and $G = \lim_{n\to\infty} G_n$.

   (a) Show that if $x \in [a, b]$ is not an endpoint of a subdivision interval and $G(x) = g(x)$ then f is continuous at x.

(b) Conversely, show that if $x \in [a, b]$ is not an endpoint of a subdivision interval and f is continuous at x then $G(x) = g(x)$.

(c) Deduce that f is Riemann integrable iff it is measurable and the set of points where f is not continuous has Lebesgue measure 0.

See also exercise 2.23 in Folland. Thus the indicator function $1_C : [a, b] \to \mathbb{R}$ of a Cantor set $C \subset [0, 1]$ is Riemann integrable iff $m(C) = 0$. Do not confuse "f is continuous on $[0, 1] \setminus E$" with "the set of discontinuities of $f : [0, 1] \to \mathbb{R}$ is E". E.g. $1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ is discontinuous at every point, but f is continuous on $\mathbb{R} \setminus \mathbb{Q}$.

5. Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = e^{-x}$. Show that f is Lebesgue integrable and compute the integral. Hint: You are allowed to compute Riemann integrals using calculus and the fact that Riemann integrable functions are Lebesgue integrable with the same integral *on finite closed intervals* as in Problem 4.. You'll have to use the monotone convergence theorem.

6. Compute
$$\lim_{n\to\infty} \int_0^\infty \left(1 + \left(\frac{x}{n}\right)\right)^{-n} \sin\left(\frac{x}{n}\right) dx$$
and justify the calculation.

7. Prove the following generalization of the Dominated Convergence Theorem. Suppose $f_n, g_n, f, g \in L^+$, $f_n \to f$, $g_n \to g$, $f_n \leq g_n$, $\int g_n \to \int g < \infty$. Then $\int f_n \to \int f$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f : X \to X$ be a measure preserving bijection, in the sense that it induces a bijection of $\mathcal{M}$ and $\mu(A) = \mu(f(A))$ for every $A \in \mathcal{M}$.

(a) Prove that for every $E \in \mathcal{M}$
$$\{x \in E \mid f^n(x) \notin E \text{ for all } n > 0\}$$
has measure 0. This is a version of *Poincaré recurrence*, saying that almost every point of E eventually returns to E under iteration.

(b) Prove that for any N the set
$$E_N = \{x \in E \mid f^n(x) \notin E \text{ for all } n > N\}$$

has measure 0. Deduce that almost every point of E returns to E infinitely many times. This is the (strong version of) Poincaré recurrence.

9. The following is an example of Feynman's integration method, found on the internet. For more of the background story on this method, see *Surely you are joking, Mr. Feynman!* by Richard Feynman. We want to compute

$$\int_0^\infty \frac{\sin(x)}{x}\, dx$$

Consider

$$I(t) = \int_0^\infty \frac{\sin(x)}{x} e^{-tx}\, dx$$

Then $I(\infty) = 0$ and

$$I'(t) = -\int_0^\infty \sin(x) e^{-tx}\, dx$$

by differentiating under the integral sign. This can be computed as an indefinite integral by integrating by parts twice, and we get

$$I'(t) = \left.\frac{e^{-tx}(\cos(x) + t\sin(x))}{1+t^2}\right|_0^\infty = -\frac{1}{1+t^2}$$

Then

$$I(\infty) - I(0) = \int_0^\infty I'(t)\, dt = -\arctan t|_0^\infty = -\frac{\pi}{2}$$

and so our integral is $I(0) = \pi/2$.

There are several things you should justify. One is differentiation under the integral sign. The more difficult thing is the discussion at $t = 0$. Note that $\sin(x)/x$ is not integrable on $[0, \infty)$. The integral is *improper*, i.e. by definition it is

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x}\, dx$$