Measurable functions and Integral

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Show that there is a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e. Hint: Simple, positive, general.

2. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Show that there is a set $A \subset [a, b]$ of measure 0 such that $f$ is continuous at every point of $[a, b] \setminus A$. Remark: The converse holds as well for bounded functions, but you don’t have to prove it. Hint: It’s an extension of an argument from class that Riemann integrable functions are Lebesgue integrable. See also exercise 2.23 in Folland.

3. Compute
\[
\lim_{n \to \infty} \int_0^\infty \left(1 + \left(\frac{x}{n}\right)^n\right) \sin \left(\frac{x}{n}\right) dx
\]
and justify the calculation.

4. Let $(X, \mathcal{M}, \mu)$ be given and for a fixed $f \in L^+$ define a new measure $\lambda$ on $\mathcal{M}$ by
\[
\lambda(E) = \int_E f \, d\mu
\]
(i) Prove that $\lambda$ is a measure.
(ii) Prove that for $g \in L^+$
\[
\int g \, d\lambda = \int fg \, d\mu
\]
(kind of a “change of variable” formula).

5. Prove the following generalization of the Dominated Convergence Theorem. Suppose $f_n, g_n, f, g \in L^+$, $f_n \to f$, $g_n \to g$, $f_n \leq g_n$, $\int g_n \to \int g < \infty$. Then $\int f_n \to \int f$. 

1