Measurable functions and integration

Set theory

1. Show that for any set S there is no surjective function $S \to 2^S$, where 2^S is the set of all subsets of S (including $\emptyset$ and S).

   Comments and a hint: This is to show that $2^c > c$ where c is the cardinality of the continuum, i.e. of the set of real numbers. Thus there are more Lebesgue measurable sets than Borel sets. (We proved that the cardinality of the set of Lebesgue sets is 2^c by looking at subsets of the middle thirds Cantor set, and I waved my hands through the argument that the cardinality of the set of Borel sets is c.)

   The motivation for the proof comes from the "barber paradox": in a village, a barber shaves everybody who doesn't shave himself. Does he shave himself? And both answers lead to a contradiction.

   So hint: if $f : S \to 2^S$ is a function, consider the set $A = \{s \in S \mid s \notin f(s)\}$. Show that A is not in the image of f, by assuming $A = f(s_0)$ and considering the possibilities $s_0 \in A$ and $s_0 \notin A$.

Measurable functions and integration

Several of the problems in this section are from old quals. Lebesgue measure is sometimes denoted m, or dm, or dx.

2. Let X be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subseteq X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a σ-algebra on X and that μ is a measure on $\mathcal{M}$. Describe the corresponding measurable functions and their integrals.

3. Let $f \in L^1(\mathbb{R})$ be an integrable function and define $F : \mathbb{R} \to \mathbb{R}$ by

   $$F(x) = \int_{-\infty}^{x} f \; dx$$

   Show that F is continuous.

   Hint: First prove the statement assuming that f is bounded. Then note that the measure of $f^{-1}((-\infty, -C) \cup (C, \infty))$ will be arbitrarily small if $C > 0$ is large.

4. Let $\{f_n\}, f$ be measurable functions on a measure space (X, μ) such that $f_1 \geq f_2 \geq \cdots \geq f \geq 0$

   Suppose that $\int f_n d\mu \to \int f d\mu$ and that f_1 is integrable. Prove that $f_n \to f$ almost everywhere. Show, by example, that this conclusion may be false if f_1 is not integrable.

5. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to \mathbb{R}$ is a real-valued function on X. Suppose further that $E_r := \{x \mid f(x) > r\}$ is measurable for each rational number r. Either prove the following assertion or find a counter-example: f is measurable.

6. Let $f \in L^1([0,1])$ have the property that for any measurable set A with $m(A) = \frac{1}{\pi}$ we have $\int_A f \; dm = 0$. Show that $f = 0$ a.e.

7. Suppose $f_i : \mathbb{R} \to \mathbb{R}$ is measurable for $i = 1, 2, \cdots$. Is the set

   $$\{x \in \mathbb{R} \mid 2 \text{ is a limit point of } \{f_i(x)\}_{i=1}^{\infty}\}$$

   measurable? A number t is a limit point of a sequence if some subsequence converges to it.

8. Let $A \subset \mathbb{R}$ such that $m(\mathbb{R} \setminus A) = 0$. Show that $A + A = \mathbb{R}$, where

   $$A + A = \{a + b \mid a, b \in A\}$$

9. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(X)$. Let

   $$E_n = \{x \in X \mid |f(x)| > n\}$$

   Show that $n\mu(E_n) \to 0$ as $n \to \infty$.

10. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(X)$. For each n, set

    $$E_n = \{x \in X \mid \frac{1}{n} \leq |f(x)| \leq n\}$$

    Then each E_n is a measurable set on which $|f|$ is bounded. Prove

    (a) $\mu(E_n)$ is finite for every n.

    (b) $\lim_{n\to\infty} \int_{E_n} f(x) d\mu(x) = \int_X f(x) d\mu(x)$.

11. Let $E \subset [0,1]$ be the set of numbers whose decimal representation contains no sevens. So for example

    $$\frac{1}{4} = 0.25000..., \frac{7}{10} = 0.69999....$$

    are in E.

(a) Show that E is a Borel set.

(b) Compute the measure of E.

12. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose that μ is *nonatomic*, i.e. whenever $A \in \mathcal{M}$ and $\mu(A) > 0$ then there exists $B \in \mathcal{M}$ such that $B \subset A$ and $0 < \mu(B) < \mu(A)$.

    (a) Show that for every $\epsilon > 0$ there is $A \in \mathcal{M}$ with $0 < \mu(A) < \epsilon$.

    (b) Fix some $a \in (0, \mu(X))$ and consider the collection

    $$\mathcal{C}_a = \{A \in \mathcal{M} \mid \mu(A) < a\}$$

    Show that this set is *not* closed under unions, i.e. there exist $A_1, A_2 \in \mathcal{C}_a$ such that $A_1 \cup A_2 \notin \mathcal{C}_a$. Applying this to $a/2$ show that there is some $A \in \mathcal{M}$ with $a/2 \leq \mu(A) < a$.

    (c) Show that for every $a \in [0, \mu(X)]$ there is $A \in \mathcal{M}$ such that $\mu(A) = a$.