General measures and Lebesgue

General measures

1. Let X be a compact metric space and μ a finite Borel measure on X (meaning that μ is defined on the Borel σ-algebra and $\mu(X) < \infty$). Show that μ satisfies both inner and outer regularity for every Borel subset $A \subset X$:

   (i) $\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}$

   (ii) $\mu(A) = \inf\{\mu(U) \mid U \supseteq A \text{ is open}\}$

   Hint: Show that the set of such As that satisfy regularity is a σ-algebra.

   Comment: More generally, by a similar proof, every *Radon measure* is regular. These are measures defined on Borel sets in locally compact σ-compact metrizable spaces so that compact sets have finite measure. E.g. the Lebesgue measure on $\mathbb{R}^d$ is a Radon measure.

2. Let $X = \{0,1\}^{\mathbb{N}}$ be the set of infinite sequences of 0s and 1s. With the product topology this space is homeomorphic to the Cantor set. Prove that there is a unique Borel measure μ on X (meaning, on the Borel σ-algebra of X) such that for every $n \geq 1$ and every choice $a_1, a_2, \cdots, a_n \in \{0,1\}$ we have

   $$\mu(\{(x_1, x_2, \cdots) \in X \mid x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n\}) = \frac{1}{2^n}$$

   Hint: Use the Carathéodory's theorem but carefully descibe the algebra and the pre-measure that serve as the input. Prove it is a pre-measure.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{N})$ a measurable space and $f : X \to Y$ a measurable function (i.e. $N \in \mathcal{N}$ implies $f^{-1}(N) \in \mathcal{M}$). Prove that

   $$\nu(N) := \mu(f^{-1}(N))$$

   defines a measure on $\mathcal{N}$. This measure ν is the *pushforward* of the measure μ by f, denoted $\nu = f_*(\mu)$.

Lebesgue measure

Denote by m the Lebesgue measure.

4. Show that if $E \subset \mathbb{R}$ is Lebesgue measurable and $m(E) > 0$ then for every $\alpha \in (0,1)$ there is an interval I with $m(E \cap I) > \alpha\, m(I)$. Hint: outer regularity.

5. If $E \subset \mathbb{R}$ is Lebesgue measurable and has positive measure, show that the set
$$E - E = \{x - y \mid x, y \in E\}$$
contains an interval around 0. Hint: Use $\alpha > \frac{3}{4}$ in Problem 4 and argue that $E - E$ contains $(-\frac{m(I)}{2}, \frac{m(I)}{2})$.

6. If $E \subset \mathbb{R}$ is Lebesgue measurable and has positive measure, show that it contains elements $a < b < c$ that form an arithmetic progression, i.e. $b = \frac{a+c}{2}$. In fact, E contains arbitrarily long arithmetic progressions.

7. Construct a Borel set $A \subset [0,1]$ such that for every nondegenerate subinterval $I \subset [0,1]$ we have
$$0 < m(A \cap I) < m(I)$$
where m denotes Lebesgue measure. Hint: For a basic building block, prove that for every interval $[a,b]$ and every $\epsilon > 0$ there is a Cantor set $C \subset [a,b]$ such that: all complementary intervals have size $< \epsilon$, the measure of C satisfies $0 < m(C) < \epsilon$, and for every open interval I that intersects C we have $m(I \cap C) > 0$. The set A will be a countable union of such building blocks.

8. Let $N \subset S^1$ be the nonmeasurable set we constructed in class. Thus, for a countable subgroup $G \subset S^1$, N contains exactly one point from each coset. Show that any measurable subset of N has measure 0.

9. Show that any measurable subset of S^1 with positive measure contains a nonmeasurable subset. Hint: Try intersecting with N and its G-translates.

   Comment: There is an analogous statement for subsets of $\mathbb{R}$, or even $\mathbb{R}^d$. The group structure and a countable subgroup ($\mathbb{Q}$ in case of $\mathbb{R}$) are crucial, but it's a bit easier to work with S^1 than $\mathbb{R}$ because it has finite measure.

10. Construct a Cantor set in $\mathbb{R}^2$ with positive measure.