Algebras, σ-algebras, measures and Carathéodory

1. Let $\mathcal{M}$ be an infinite σ-algebra. Show that $\mathcal{M}$ is uncountable. Hint: First show that there are infinitely many pairwise disjoint (nonempty) elements of $\mathcal{M}$.

2. When $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing chain of sets we write $\lim_{i\to\infty} A_i := \cup_i A_i$, and similarly if $A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing chain we write $\lim_{i\to\infty} A_i := \cap_i A_i$. In the same vain, if A_i are arbitrary subsets of an ambient set we write

$$\limsup_{i\to\infty} A_i := \cap_{j=1}^{\infty} \cup_{i=j}^{\infty} A_i$$

and

$$\liminf_{i\to\infty} A_i := \cup_{j=1}^{\infty} \cap_{i=j}^{\infty} A_i$$

Let $(X, \mathcal{M}, \mu)$ be a measure space and $A_i \in \mathcal{M}$ measurable sets.

   (i) Show that $\liminf_{i\to\infty} A_i$ and $\limsup_{i\to\infty} A_i$ are measurable.

   (ii) Show that $\mu(\liminf_{i\to\infty} A_i) \leq \liminf_{i\to\infty} \mu(A_i)$.

   (iii) Assuming $\mu(X) < \infty$ show $\mu(\limsup_{i\to\infty} A_i) \geq \limsup_{i\to\infty} \mu(A_i)$.

   (iv) Prove the Borel-Cantelli lemma: Let $(X, \mathcal{M}, \mu)$ be a measure space and let A_i be measurable sets such that $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Then the set

$$\Omega = \left\{ x \in X \mid x \in A_i \text{ for infinitely many } i \right\}$$

   of points that belong to infinitely many A_i has measure 0.

3. Let μ be a Borel measure on $\mathbb{R}$ (i.e. a measure defined on the Borel σ-algebra). Suppose it satisfies:

   (i) $\mu([0,1]) = 1$.

   (ii) $\mu(A + t) = \mu(A)$ for every Borel set A and every $t \in \mathbb{R}$, i.e. μ is translation invariant.

   Show that μ equals the Lebesgue measure.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Let A be a subset of $\mathbb{R}$ of (Lebesgue) measure 0 such that f is differentiable at every $x \in A$. Show that $f(A)$ has measure 0. (Compare with Sard's theorem from the manifolds class.)

5. Let $A \subset [0,1]$ be a Lebesgue-measurable set with positive measure. Show that there are $x, x' \in A$ with $x - x' \in \mathbb{Q} \setminus \{0\}$.

6. Let
$$A = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{3^n} \mid x_n = 0 \text{ or } 1 \right\}$$
Show that A has Lebesgue measure 0 but $A + A := \{x + y \mid x, y \in A\} = [0,1]$.

7. Let $\mathcal{A}$ be the collection consisting of finite unions of sets of the form $(a,b] \cap \mathbb{Q}$, $-\infty \leq a < b \leq \infty$ (including $\emptyset$). Show that
   (i) $\mathcal{A}$ is an algebra.
   (ii) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.
   (iii) Define $\rho : \mathcal{A} \to [0,\infty]$ by $\rho(\emptyset) = 0$ and $\rho(A) = \infty$ when $A \neq \emptyset$. Show that ρ is a pre-measure (in addition to finite additivity and $\rho(\emptyset) = 0$, $\rho(\sqcup_i A_i) = \sum_i \rho(A_i)$ whenever the infinite union is in $\mathcal{A}$) that has more than one extension to a measure on $\mathcal{P}(\mathbb{Q})$. Explain why this doesn't contradict Carathéodory's theorem.

8. Recall that an abstract outer measure is a function $\mu^* : \mathcal{P}(X) \to [0,\infty)$ satisfying
   (i) $\mu^*(\emptyset) = 0$,
   (ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$, and
   (iii) $\mu^*(\cup_i^\infty A_i) \leq \sum_i^\infty \mu^*(A_i)$.

   Show by example that (ii) does not follow from (i) and (iii).

9. Let $\mathcal{E} \subset \mathcal{P}(X)$ be an algebra of sets and $\rho : \mathcal{E} \to [0,\infty)$ an additive measure, as in the setting of Carathéodory's theorem. Let $\mu^* : \mathcal{P}(X) \to [0,\infty)$ be the associated outer measure. Recall that in class we defined a set $M \subseteq X$ to be *measurable* if for every $\epsilon > 0$ there exists some $E \in \mathcal{E}$ such that $\mu^*(M \triangle E) < \epsilon$. Show that M is measurable if and only if it satisfies the *Carathéodory condition*:

For every subset $P \subseteq X$

$$\mu^*(P) = \mu^*(P \cap M) + \mu^*(P \smallsetminus M)$$

Comment: In general, μ^* is not additive, but this says that when we cut by a measurable set it is. The advantage of the Carathéodory condition is that one doesn't need $\mathcal{E}$, only an abstract outer measure, to talk about measurable sets. You can look up in Folland the proof that even in this more general setting the collection of measurable sets is a σ-algebra and μ^* is a measure on it.