Stone-Weierstrass and Fourier transform

Stone-Weierstrass

1. Let X be compact Hausdorff and $\mathcal{A} \subset C(X)$ a closed subalgebra. Suppose that $\mathcal{A}$ separates point (but don't assume that $\mathcal{A}$ contains constants). Show that either

   (i) $\mathcal{A} = C(X)$, or

   (ii) there exists $x_0 \in X$ so that

   $$\mathcal{A} = \{f \in C(X) \mid f(x_0) = 0\}$$

   Subalgebras as in (ii) are maximal ideals in $C(X)$.

2. Let X be a compact metric space with distance function d and a basepoint x_0. Suppose Z is a countable dense set. Let $\mathcal{B}$ be the rational vector space spanned by functions $f_z : X \to \mathbb{R}$ defined by $f_z(x) = d(x, z)$ and by the constant function 1, and finite products of these functions. Then $\mathcal{B}$ is countable. Show that it is dense in $C(X)$, thus showing that $C(X)$ is separable when X is.

Fourier analysis

All of the problems below except Problem 4 are from the old quals.

3. Suppose $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Show that $f \in L^2(\mathbb{R})$. Hint: By Riemann-Lebesgue $\hat{f} \in L^2(\mathbb{R})$.

4. By considering $f = \mathbb{1}_{[-1,1]}$ use the Fourier transform to show

   $$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt = \pi$$

5. Show that the Fourier transform of $\mathbb{1}_{(0,1)}$ is not in $L^1(\mathbb{R})$.

6. Show that the Fourier transform is 1-1 on $L^1(\mathbb{R})$. Hint: Problem 3 plus Plancherel.

7. If $f, g \in L^1(\mathbb{R})$ then the convolution $f * g \in L^1(\mathbb{R})$ and $\widehat{f * g} = \hat{f}\hat{g}$.

8. Let $f \in L^1(S^1)$. Show that $\int_{S^1} f(t)e^{-2\pi n t}\, dt \to 0$ as $n \to \infty$ (Riemann-Lebesgue for Fourier series).