Hilbert spaces

$\mathcal{H}$ will denote a Hilbert space.

1. Show that the Parallelogram Law fails in $L^p([0,1])$ for $p \neq 2$.

2. Let $e_1, e_2, \cdots$ be an orthonormal basis of a Hilbert space $\mathcal{H}$.

   (i) Show that 0 is the weak limit of the sequence $e_1, e_2, \cdots$.

   (ii) $B = \{x \in \mathcal{H} \mid ||x|| \leqslant 1\}$ is closed in the weak topology.

   (iii) $S = \{x \in \mathcal{H} \mid ||x|| = 1\}$ is dense in B.

3. Prove that the set of $T \in L(\mathcal{H}, \mathcal{H})$ which are isometries is a closed set in the strong operator topology but not necessarily in the weak operator topology. Here an *isometry* is an operator T such that $\|Tx\| = \|x\|$ for every x, but T is not required to be onto. There are examples where $T_n \to T$ strongly, with T_n unitary (i.e. isometry onto) but T is not onto. (You may want to try to find one.)

4. Let $T \in L(\mathcal{H}, \mathcal{H})$.

   (i) There is a unique linear $T^* : \mathcal{H} \to \mathcal{H}$ such that

   $$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

   for all $x, y \in \mathcal{H}$ (Hint: Riesz)

   (ii) $\|T^*\| = \|T\|$ and $T^{**} = T$.

   (iii) $Im(T)^\perp = Ker(T^*)$ and $Ker(T)^\perp = \overline{Im(T^*)}$ (the "bar" means closure here).

   (iv) For $\mathcal{H} = \mathbb{C}^n$, identifying T with a complex $n \times n$ matrix in the usual way, T^* is the conjugate-transpose of T.

5. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and define $T_n : \mathcal{H} \to \mathcal{H}$ by $T_n(e_i) = e_{n+i}$ for all $i \in \mathbb{N}$. Let $T_n^* : \mathcal{H} \to \mathcal{H}$ be the adjoint of T_n.

   (i) Show that $T_n^*(e_i) = e_{i-n}$ if $i > n$ and 0 if $i \leqslant n$.

   (ii) The sequence T_n does not converge in the strong operator topology, and $T_n \to 0$ in the weak operator topology.

   (iii) $T_n^* \to 0$ in the strong operator topology.

Information: Adjoint, viewed as a map $L(\mathcal{H},\mathcal{H}) \to L(\mathcal{H},\mathcal{H})$, is continuous in the weak operator topology, but not in the strong operator topology unless $\dim \mathcal{H} < \infty$.

6. Let $x_n \in \mathcal{H}$ be a sequence in a Hilbert space. Assume that for every $y \in \mathcal{H}$ the sequence $\langle x_n, y \rangle$ in $\mathbb{R}$ converges. Show that there is some $x \in \mathcal{H}$ so that $\langle x_n, y \rangle \to \langle x, y \rangle$ for every $y \in \mathcal{H}$.

7. Let $M \subset \mathcal{H}$ be a linear subspace and $f : M \to \mathbb{C}$ a bounded linear functional. Show that there is a unique extension of f to a linear functional $F : \mathcal{H} \to \mathbb{C}$ such that $\|F\| = \|f\|$. Also show that for this F we have $F|M^\perp = 0$.

8. Let $\alpha : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Show that

$$A(f)(x) = \int_0^1 \alpha(x,y) f(y)\, dy$$

defines a bounded operator $L^2([0,1]) \to C([0,1])$.

9. Let $P : \mathcal{H} \to \mathcal{H}$ be a self-adjoint ($\langle Px, y \rangle = \langle x, Py \rangle$) idempotent ($P^2 = P$). Show that P is the orthogonal projection to a closed linear subspace. We are not assuming P is bounded.