Weak and Weak-* Topology

Nets

A primer on nets. A *directed set* is a partially ordered set $(I, \leqslant)$ such that for all $i, j \in I$ there is $k \in I$ with $i \leqslant k, j \leqslant k$.

Examples: $I = \mathbb{N}$ with the usual ordering. The set I of open sets in a topological space Ω containing a given point x, and ordering $U \leqslant V$ when $V \subseteq U$.

Let Ω be a topological space. A *net* in X is a function $\phi : I \to \Omega$ for a directed set I. Instead of $\phi(i)$ we write x_i. We say that the net (x_i) converges to $x \in \Omega$, and we write $x_i \to x$, if for every neighborhood U of x there is some $i \in I$ such that $i \leqslant j$ implies $x_j \in U$. If $Z \subset \Omega$ is closed and $x_i \in Z$ then the limit (if it exists) of the net (x_i) belongs to Z.

1. This problem illustrates the need to use nets when discussing weak topologies. Let
$$S = \{\sqrt{n} e_n \mid n = 1, 2, 3, \cdots\} \subset \ell^2$$
where $e_n \in \ell^2$ is the element with n^{th} coordinate 1 and other coordinates 0.
   (a) Show that there is no sequence in S that converges to 0 in the weak topology. Hint: Problem 4.
   (b) Show that 0 is in the weak closure of S (by the "weak closure" I mean closure in the weak topology). That is, show that every weak neighborhood of 0 contains (infinitely many) elements of S.

2. In the setting of Problem 1. show that there is a net in S converging to 0. More generally, show that in any topological space Ω if x is in the closure of a set S then there is a net in S converging to x. Hint: Use a system of neighborhoods of x for the index set.

Alaoglu

3. Let $\mathcal{F} = \{f_\alpha \mid \alpha \in \mathcal{A}\} \subset C(X)$ be a family of continuous functions on a compact metrizable space X. Let $c_\alpha \in \mathbb{R}$ be a real number for each $\alpha \in \mathcal{A}$. Suppose that for every finite subcollection $F \subset \mathcal{F}$ there is a signed Borel measure μ_F such that $|\mu_F|(X) \leqslant 1$ and $\int f_\alpha \, d\mu_F = c_\alpha$ for all $f_\alpha \in F$. Show that there is a signed Borel measure μ such that $\int f_\alpha \, d\mu = c_\alpha$ for every $f_\alpha \in F$.

Weak topology

4. Suppose $x_n \xrightarrow{w} x$ weakly in a Banach space V. (Recall that this means that $f(x_n) \to f(x)$ for every $f \in V^*$.) Show that the norms $\|x_n\|$ are uniformly bounded. Hint: View x_n as functionals on V^* and apply Banach-Steinhaus.

5. If $x_n \xrightarrow{w} x$ then $\|x\| \leqslant \liminf_{n\to\infty} \|x_n\|$. Hint: Use Hahn-Banach to find $f \in V^*$ so that $\|f\| = 1$ and $f(x) = \|x\|$.

Strong and weak topologies on $L(V, W)$

Let V, W be Banach spaces. There are topologies on the space $L(V, W)$ of bounded operators $V \to W$ that are weaker than the operator norm topology. The *strong operator topology* on $L(V, W)$ is characterized by $T_n \xrightarrow{s} T$ if and only if $T_n(x) \to T(x)$ in the norm topology of W for every $x \in V$. In spite of its name, this is weaker than the operator norm topology. As you have already guessed, we should really be using nets here. Similarly, the *weak operator topology* is characterized by $T_n \xrightarrow{w} T$ if and only if $T_n(x) \to T(x)$ in the weak topology of W for every $x \in V$. This topology is even weaker than the strong topology.

6. Consider the case $V = \mathbb{R}$. The space $L(\mathbb{R}, W)$ is canonicaly isomorphic to W via $T \mapsto T(1)$.
   (a) The norm topology on $L(\mathbb{R}, W) = W$ coincides with the norm topology on W.
   (b) The strong topology on $L(\mathbb{R}, W) = W$ coincides with the norm topology on W.
   (c) The weak topology on $L(\mathbb{R}, W) = W$ coincides with the weak topology on W.

7. Consider the case $W = \mathbb{R}$, thus $L(V, \mathbb{R}) = V^*$.
   (a) The norm topology on $L(V, \mathbb{R}) = V^*$ coincides with the norm topology on V^*.
   (b) The strong topology on $L(V, \mathbb{R}) = V^*$ coincides with the weak-* topology on V^*.
   (c) The weak topology on $L(V, \mathbb{R}) = V^*$ coincides with the weak-* topology on V^*.

The Portmanteau theorem[1]

Recall from class:

Theorem (Portmanteau)**.** *Let X be compact metrizable. The following are equivalent for probability Borel measures μ_n, μ on X.*

(i) $\mu_n \overset{w*}{\to} \mu$ *in weak-* topology.*

(ii) For every closed set $F \subset X$ we have

$$\limsup_{n\to\infty} \mu_n(F) \leqslant \mu(F)$$

(iii) For every open set $U \subset X$

$$\liminf_{n\to\infty} \mu_n(U) \geqslant \mu(U)$$

(ii) and (iii) are equivalent by taking complements.

8. Show that (i) implies (ii). Hint: I described this briefly in class. Let $f : X \to [0,1]$ be continuous with $f = 1$ on F. Consider $\int f \, d\mu_n \to \int f \, d\mu$ and arrange that the right hand side is within ϵ of $\mu(F)$.

9. Show that (ii) implies (i). Hint: By (ii) and (iii) $\mu(F) = \mu(int(F)) = \lim_{n\to\infty} \mu_n(F)$ when $\mu(\partial F) = 0$. Now use Problem 8 from the Fubini homework (the Distribution theorem): for $f : X \to [0,\infty)$ we have $\int_X f \, d\mu = \int_0^\infty \mu(\{f \geqslant t\}) \, dt$ and similarly for μ_n. The sets $\{f \geqslant t\}$ will satisfy the first sentence for all but countably many t.

[1] A portmanteau is a large suitcase or a travel trunk in French. The portmanteau theorem is a collection of many (wikipedia lists 7 but I think there are more) equivalent statements of which I am presenting only three. I would need a portmanteau to bring to you all of them.