Open Mapping, Closed Graph, Banach-Steinhaus, Riesz

Closed Graph Theorem

1. Let A, B, C be Banach spaces, $h : A \to B$, $g : B \to C$, $f = gh : A \to C$ linear functions, with g and f bounded. If g is 1-1, show that h is bounded.

2. Let $T : V \to W$ be a linear map between Banach spaces. Let $\mathcal{A} \subset W^*$ be a set of bounded functionals on W that *separate points*, i.e. for all $w, w' \in W$ with $w \neq w'$ there is $\phi \in \mathcal{A}$ so that $\phi(w) \neq \phi(w')$. Assume that compositions $\phi T : V \to \mathbb{R}$ are bounded for all $\phi \in \mathcal{A}$. Show that T is bounded. Hint: In class we did the case $\mathcal{A} = W^*$.

3. Let V be a Banach space and $T : V \to C([0, 1])$ a linear operator such that $x_n \to 0$ in V imples that $T(x_n) \to 0$ pointwise. Show that T is bounded. Hint: Use Problem 2. Try evaluations at points of $[0, 1]$.

Open Mapping Theorem

4. Let $T : V \to W$ be a surjective bounded operator between Banach spaces. Assume that T sends the closed unit ball to a compact set. Show that $\dim W < \infty$.

Banach-Steinhaus

5. Let V, W be Banach spaces, $T_i : V \to W$ a sequence of bounded linear operators such that $\lim_{i\to\infty} T_i v$ exists for every $v \in V$. Define the linear map $T : V \to W$ by

$$Tv = \lim_{i\to\infty} T_i v$$

Show that T is bounded.

6. Let $f_n \in C([0, 1])$ for $n = 1, 2, \cdots$. Show that the following two statements are equivalent:

   (a) for every $\lambda \in C([0, 1])^*$ we have $\lambda(f_n) \to 0$ as $n \to \infty$,

   (b) $f_n(x) \to 0$ for every $x \in [0, 1]$ and $\sup ||f_n||_\infty < \infty$.

   Hint: This also uses Riesz representation theorem.

7. Let V be a Banach space and $X \subset V$ a subset. Suppose that for every $f \in V^*$ the set $f(X) \subset \mathbb{R}$ is bounded. Show that X is a bounded subset of V.

8. Let $f_n \in L^1(\mathbb{R})$ and assume that $\lim_{n\to\infty} \int f_n g = 0$ for every $g \in L^\infty(\mathbb{R})$. Show that the sequence f_n is bounded in $L^1(\mathbb{R})$.

Riesz representation theorem

A couple of questions from the old quals.

9. Let X be a compact Hausdorff space and $A \subset C(X)$ a linear subspace. Let $S \subset X$ be a closed subspace. Suppose A has the following two properties:

   - A contains constant functions.
   - For every $f \in A$

   $$\sup\{|f(x)| \mid x \in X\} = \sup\{|f(x)| \mid x \in S\}$$

   Show that for every $x \in X$ there is a positive Borel measure μ_x on S such that for every $f \in A$

   $$f(x) = \int_S f \, d\mu_x$$

   Hint: The restriction $C(X) \to C(S)$ is injective on A and if the restriction of $f \in A$ is $\geqslant 0$ on S then $f \geqslant 0$ on X.

10. Let X be a compact Hausdorff space and $H \subset C(X)$ a linear subspace. Assume that if $f \in H$ and $f \geqslant 0$ then $f = 0$. Show that there is a positive Borel measure μ on X such that $\mu(X) = 1$ and $\int h \, d\mu = 0$ for every $h \in H$.

    Hint: Using the positive cone version of Hahn-Banach find a functional that sends constant 1 to 1 and is 0 on H.

Some norm computations

11. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping represented by the $m \times n$ matrix (a_{ij}) with respect to the standard bases in $\mathbb{R}^n$, $\mathbb{R}^m$. Compute the norm of T if:

(a) $\mathbb{R}^n$ is equipped with the ℓ^1 norm and $\mathbb{R}^m$ with the ℓ^∞ norm.

(b) $\mathbb{R}^n$ is endowed with the ℓ^∞ norm and $\mathbb{R}^m$ with the ℓ^1 norm.

(c) $\mathbb{R}^n$ and $\mathbb{R}^m$ are both equipped with the ℓ^1 norm.

(d) $\mathbb{R}^n$ and $\mathbb{R}^m$ are both endowed with the ℓ^∞ norm.

12. Let T be the mapping from sequences to sequences given by $T(x) = y$ where $y_n = 2^{-n} \sum_{k=1}^{n} x_k$. Prove that T is bounded from ℓ^∞ to ℓ^1 and compute its norm.