The Hahn-Banach Theorem and Banach pathologies galore

1. Prove that $L^p(\mathbb{R}^n)$ is separable for $1 \leqslant p < \infty$. Hint: Show that rational linear combinations of characteristic functions of boxes with rational coordinates are dense. More generally, $L^p(\mu)$ is separable for $1 \leqslant p < \infty$ when the ring of sets used to define μ via Carathéodory is countable. Recall that ℓ^∞ is not separable, and neither is $L^\infty(\mathbb{R}^n)$. Also recall that for separable Banach spaces, Hahn-Banach can be proved without the axiom of choice.

2. Prove that continuous functions of compact support are dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.

3. Prove the following version of Hahn-Banach (the *Positive Cone* version). Let V be a real vector space and $P \subset V$ a *convex cone* i.e. $x, y \in P$, $a, b \geqslant 0$ imply $ax + by \in P$. Let $W \subset V$ be a subspace such that for every $v \in V$ there is $w \in W$ with $w - v \in P$. (We think of P as being the set of positive vectors and we write $w \geqslant v$.) Let $f : W \to \mathbb{R}$ be a functional which is *positive* i.e. $f(w) \geqslant 0$ for $w \in W \cap P$. Show that f extends to a functional $\tilde{f} : V \to \mathbb{R}$ which is also positive, i.e. $\tilde{f}(v) \geqslant 0$ if $v \in P$.

   Hint: When extending from W to $W + \mathbb{R}x$ set $f(x) = \alpha$ where α is chosen so that
   $$f(W^{\leqslant x}) \leqslant \alpha \leqslant f(W^{\geqslant x})$$
   where
   $$W^{\leqslant x} = \{w \in W \mid w \leqslant x\}, \qquad W^{\geqslant x} = \{w \in W \mid w \geqslant x\}$$
   both of which are nonempty by assumption (why is $W^{\leqslant x}$ nonempty?). Argue that such a choice is possible.

4. Let V be a Banach space and $A \subset V$ a closed subspace. If $x \in V \smallsetminus A$ show that $span(A \cup \{x\})$ is a closed subspace. Hint: Consider $f \in V^*$ so that $f(x) \neq 0$ but $f|A = 0$.

5. The following exercise shows that if A, B are closed subspaces of a Banach space, $A + B$ may not be closed. Exercise 4. shows that it is closed if $\dim B < \infty$.

(a) Let $c_0 \subset \ell^\infty$ be the closed subspace consisting of sequences $(x_i)_{i=1}^\infty$ such that $\lim x_i = 0$. So c_0 is a Banach space with the ℓ^∞-norm. Let

$$A = \{(x_i) \mid x_{2n} = 0 \text{ for all } n\}$$

and

$$B = \{(x_i) \mid x_{2n} = \frac{x_{2n-1}}{n} \text{ for all } n\}$$

Show that A and B are closed subspaces of c_0. Hint: E.g. write A as the intersection of kernels of $(x_i) \to x_{2n}$ but argue that such functionals are bounded.

(b) Show that $A + B$ is exactly the set of sequences (x_i) such that $nx_{2n} \to 0$ for all n.

(c) Deduce that $A+B$ is dense in c_0 but that it is a proper subspace.

6. Let $V = C[-1, 1]$ (continuous functions with the sup norm). Let $W \subset V$ be the subspace consisting of functions f with $\int_{-1}^0 f = \int_0^1 f = 0$. Let $g \in V$ be a function such that $\int_{-1}^0 g = -1$ and $\int_0^1 g = 1$. Compute $d(g, W)$ and show that it is not realized by any $f \in W$.

7. Is there a norm on $\mathbb{R}^2$ such that $\|(1, 0)\| = \|(0, 1)\| = 1$ and $\|(1, 1)\| < 1$?

8. Prove the Riesz lemma: Let V be a Banach space, and $W \subset V$ a proper closed subspace. Then for every $a < 1$ there is a vector $v \in V$ such that $\|v\| = 1$ and $d(v, W) > a$.

9. Show that the Riesz lemma cannot be improved to $d(v, W) = 1$. That is, find an example of a Banach space V and a proper closed subspace $W \subset V$ so that for every $v \in V$ with $\|v\| = 1$ necessarily $d(v, W) < 1$. Hint: Problem 6.

10. Show that finite dimensional subspaces of Banach spaces are closed.

11. Let V be an infinite dimensional Banach space. Using Problems 8. and 10. inductively construct a sequence $v_1, v_2, \cdots$ so that $\|v_i\| = 1$ and $d(v_i, v_j) > 0.99$ for $i \neq j$. Deduce that the unit sphere in V is not compact.

For the end, a fun Baire category problem.

12. Let $f : (0, \infty) \to (0, \infty)$ be a continuous function with the property that for every $x > 0$ we have

$$\lim_{n\to\infty} f(nx) = 0$$

Show that $\lim_{x\to\infty} f(x) = 0$. Hint: For $\epsilon > 0$ consider sets $B_N = \{x \in (0, \infty) \mid |f(nx)| \leqslant \epsilon \text{ for } n \geqslant N\}$.