Jordan measure

1. Let $m' : \mathcal{E} \to [0, \infty)$ be a function defined on the elementary sets in $\mathbb{R}^d$ satisfying:

   (i) (finite additivity) $m'(A \sqcup B) = m'(A) + m'(B)$ for disjoint $A, B \in \mathcal{E}$,

   (ii) (translation invariance) $m'(A + x) = m'(A)$ for $A \in \mathcal{E}$, $x \in \mathbb{R}^d$,

   (iii) (normalization) $m'([0,1)^d) = 1$.

   Show that $m' = m$ is the standard measure on elementary sets. How do you modify the proof if normalization is stated as $m'([0,1]^d) = 1$?

   Hint: First show that $m' = m$ on half-open $\frac{1}{N_1} \times \frac{1}{N_2} \times \cdots \times \frac{1}{N_d}$-boxes.

2. Let m' be a function defined on Jordan measurable sets in $\mathbb{R}^d$ with values in $[0, \infty)$ satisfying finite additivity, translation invariance and normalization. Then $m' = m$ agrees with the Jordan measure.

3. Let $f : [a, b] \to [0, \infty)$ be a continuous function. Show that the set

   $$A = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], 0 \leq y \leq f(x)\} \subset \mathbb{R}^2$$

   is Jordan measurable. Also show that the graph

   $$\Gamma = \{(x, f(x)) \mid x \in [a, b]\}$$

   of f is Jordan measurable and has measure 0.

   Hint: f is uniformly continuous.

4. Show that any

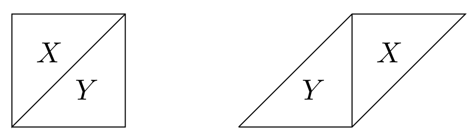
   - round disk
   - triangle
   - convex polygon

   in $\mathbb{R}^2$ is Jordan measurable. Can you generalize to any polygon? Hint: You can argue by induction on the number of sides, but you should show that if the number of sides is > 3 there is always an internal or an external diagonal, so that the polygon can be written as a (essentially) disjoint union or difference of two simpler polygons.

5. Let $A \subset \mathbb{R}^d$ such that for every $\epsilon > 0$ there are Jordan measurable sets B, C with $B \subset A \subset C$ such that $m(C) - m(B) < \epsilon$. Show that A is Jordan measurable. This corresponds more closely to the ancient Greeks' method of inscribing and circumscribing polygons to estimate area.

6. (This one is more rewarding – read "harder" – than the others.) Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map. Show that if $A \subset \mathbb{R}^d$ is Jordan measurable so is $L(A)$ and
$$m(L(A)) = |\det L|\; m(A)$$

   Discussion: It suffices to prove the statement for elementary matrices. The first step is to show this when A is a box. If L is diagonal $L(A)$ is also a box, and otherwise the idea is to show that A and $L(A)$ are "cut and paste" equivalent, as suggested by the figure below, which suggests how to compute $m(L(A))$ when $A = [0,1]^2$ and $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.



   The next step is to consider the function $m'(A) = m(L(A))$ and use Problems 1-2. There will be a simpler argument for the first step when we learn about Lebesgue measure and Fubini's theorem.

   Another approach is more algebraic. There is a continuous homomorphism $\phi : GL_d(\mathbb{R}) \to \mathbb{R}^+$ so that $m(L(A)) = \phi(L)m(A)$. An algebraic fact is that every homomorphism $SL_d(\mathbb{R}) \to \mathbb{R}^+$ is trivial. This implies that $\phi(L) = |\det L|^r$ for some $r > 0$. To see that $r = 1$ consider diagonal matrices.

7. Let $A \subset \mathbb{R}^d$ be Jordan measurable. Show that both the interior $int(A)$ and closure $\overline{A}$ are Jordan measurable and have the same Jordan measure as A.

8. Show that $[0,1] \cap \mathbb{Q} \subset \mathbb{R}$ and $[0,1]^2 \cap \mathbb{Q}^2 \subset \mathbb{R}^2$ are not Jordan measurable.

9. (Another more rewarding problem.) Construct a bounded open set in $\mathbb{R}$ that is not Jordan measurable. Construct a compact set in $\mathbb{R}$ that is not Jordan measurable.

10. (Carathéodory) Let $B \subset \mathbb{R}^d$ be a bounded set and $A \subset \mathbb{R}^d$ a Jordan measurable set. Show that

$$m^{*,J}(B) = m^{*,J}(B \cap A) + m^{*,J}(B \setminus A)$$

where $m^{*,J}$ denotes outer Jordan measure.

Comment: We know that there are ways of decomposing a set as a disjoint union so that the sum of outer measures of the pieces is strictly greater than the outer measure of the original set (see e.g. Problem 8.). The problem is to show that this doesn't happen when the "cut" is induced by a Jordan measurable set. Later, we will see that the corresponding statement for Lebesgue measure *characterizes* measurable sets.

Riemann and Darboux integrals

11. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable iff it is Darboux integrable and in that case the two integrals coincide.

12. Show that every continuous function $f : [a, b] \to \mathbb{R}$ is Darboux integrable. Hint: Uniform continuity.

13. Riemann integral is a linear and monotone function on the space of Riemann integrable functions on $[a, b]$, and $\int_a^b 1_A = m(A)$ when A is Jordan measurable.

14. Show that properties in Problem 13 uniquely determine the Riemann integral.

15. Let $f : [a, b] \to [0, \infty)$ be a bounded nonnegative function. Then f is Riemann integrable iff the set

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], 0 \leq y \leq f(x)\}$$

is Jordan measurable, and if so then $\int_a^b f = m(A)$.