Math 5520 Extra credit Homework

An application of the ping-pong lemma

The purpose of this exercise is to show that the homeomorphism group of the real line contains the free group F_2 as a subgroup.

Define the homeomorphism $f : [0,1] \to [0,1]$ by

$$f(t) = \begin{cases} 4t, & t \in [0, \frac{1}{5}] \\ \frac{4}{5} + \frac{1}{4}(t - \frac{1}{5}), & t \in [\frac{1}{5}, 1] \end{cases}$$

Exercise 1. Show that

$$f^n\left(\left[\frac{1}{5}, 1\right]\right) \subseteq \left[\frac{4}{5}, 1\right]$$

for $n = 1, 2, \cdots$ and

$$f^n\left(\left[0, \frac{4}{5}\right]\right) \subseteq \left[0, \frac{1}{5}\right]$$

for $n = -1, -2, \cdots$.

Then define homeomorphisms $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$ by

$$\gamma_1(t) = [t] + f(\{t\})$$

where $[t]$ is the largest integer $\leq t$ and $\{t\} = t - [t]$ is its fractional part, and

$$\gamma_2(t) = T\gamma_1 T^{-1}$$

where $T(t) = t - \frac{1}{2}$. Thus γ_2 is a version of γ_1 translated by $\frac{1}{2}$.

Also define

$$X_1 = \cup_{k \in \mathbb{Z}} \left[k - \frac{1}{5}, k + \frac{1}{5}\right]$$

and $X_2 = T(X_1)$.

Exercise 2. Show that $\gamma_1^n(X_2) \subseteq X_1$ and $\gamma_2^n(X_1) \subseteq X_2$ for $n \neq 0$.

Exercise 3. Use ping-pong to argue that γ_1, γ_2 generate a free subgroup of the homeomorphism group $Homeo(\mathbb{R})$ of the real line.