Hyperbolic Geometry Homework, due March 29

This homework is more computational than the previous. You should get started early.

1. Show that in upper half space model the distance between points $z$ and $w$ is

$$D(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}$$

Hint: In class we computed the distance between $i$ and $yi$ for $y > 1$. Show that the formula above gives the same answer. Then show that $D(f(z), f(w)) = D(z, w)$ when $f$ is a horizontal translation $z \mapsto z + c$ for $c \in \mathbb{R}$, a dilation $z \mapsto \lambda z$, $\lambda > 0$, or the inversion in the unit circle $z \mapsto 1/\bar{z}$. Why does that complete the proof?

2. Show that in upper half space model rotation by angle $\alpha$ about $i$ is given by

$$z \mapsto \frac{z \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}}{-z \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}}$$

Hint: Show that $i \mapsto i$ and compute the derivative at $i$.

3. Sketch the triangle in the upper half space model with vertices 0, $\infty$ and $1 + i$ and compute its area.

4. Show that the circumference of a hyperbolic circle of radius $R$ is $2\pi \sinh R$ and the area is $4\pi \sinh^2 \frac{R}{2}$.

Hint: Work in the disk model. Recall that the metric at distance $r$ from $O$ is given by dividing Euclidean length by $\frac{1 - r^2}{2}$ (in class we ignored the factor 2, but to make it consistent with upper half space distance it should really be there). Center the circle at $O$ and show that Euclidean radius $r$ is related to $R$ via $R = \log \frac{1 + r}{1 - r}$. First work out the circumference and then integrate. You can look up the required integrals.