Math 5520 Homework 4

The Seifert-van Kampen Theorem

Most of the problems come from *Hatcher: Algebraic Topology.* For a full credit, solve 4 of the 6 problems. Do all 6 for extra credit!

1. Let $U_1, U_2, \cdots, U_m$ be open convex subsets of $\mathbb{R}^n$ such that all triple intersections $U_i \cap U_j \cap U_k$ are nonempty. Show that $X = U_1 \cup U_2 \cup \cdots \cup U_m$ is simply connected.
2. Show that if U is an open (path-)connected subset of $\mathbb{R}^3$ and $p \in U$, then $\pi_1(U \setminus \{p\})$ is isomorphic to $\pi_1(U)$. Deduce that $\mathbb{R}^3$ with finitely many points removed is simply-connected.
3. Let X be the complement in $\mathbb{R}^3$ of n straight lines through the origin. Compute $\pi_1(X)$.
4. Let X be the space obtained from the 2-sphere by identifying the north and the south poles. Put a cell complex structure on X and compute $\pi_1(X)$.
5. Compute the fundamental group of the space obtained from the disjoint union of two tori $S^1 \times S^1$ by identifying the circle $S^1 \times \{x\}$ on one torus with $S^1 \times \{x\}$ on the other.
6. Show that $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ is uncountable.

Group Presentations

For a full credit, solve one of the following two problems. Do both for extra credit!

7. The *dihedral group* D_4 of order 8 is the group of symmetries of the square $[-1, 1]^2$. It is generated by the reflection a in the line $x = 0$ and the reflection b in the line $y = x$. Draw the Cayley graph and find a presentation of D_4 by figuring out which 2-cells need to be attached to make a simply-connected 2-complex.

8. The *quaternion group* is the group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$. Draw the Cayley graph of Q with respect to the generators i, j. Find relations so that gluing 2-cells along them produces a simply connected 2-complex. Deduce a presentation for Q. This problem is harder.

There is a picture of the Cayley graph for Q on Wikipedia. There is also one for D_4, but watch out – their generators are different from mine.