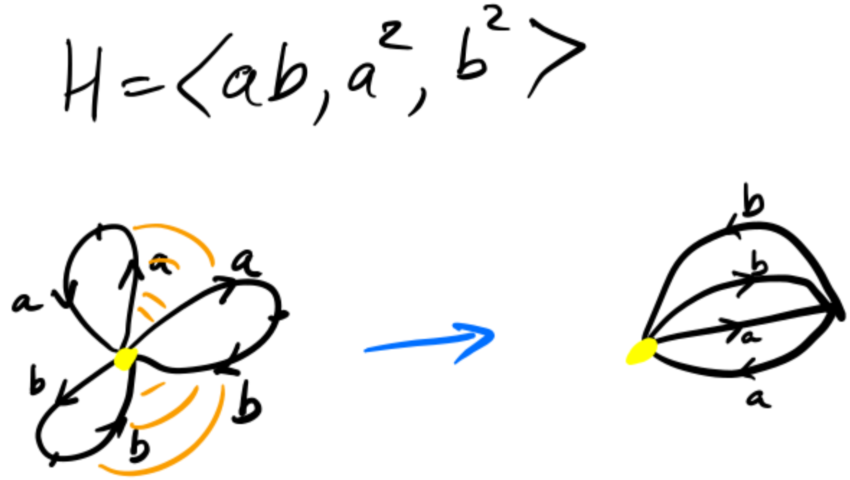
Summary of folding

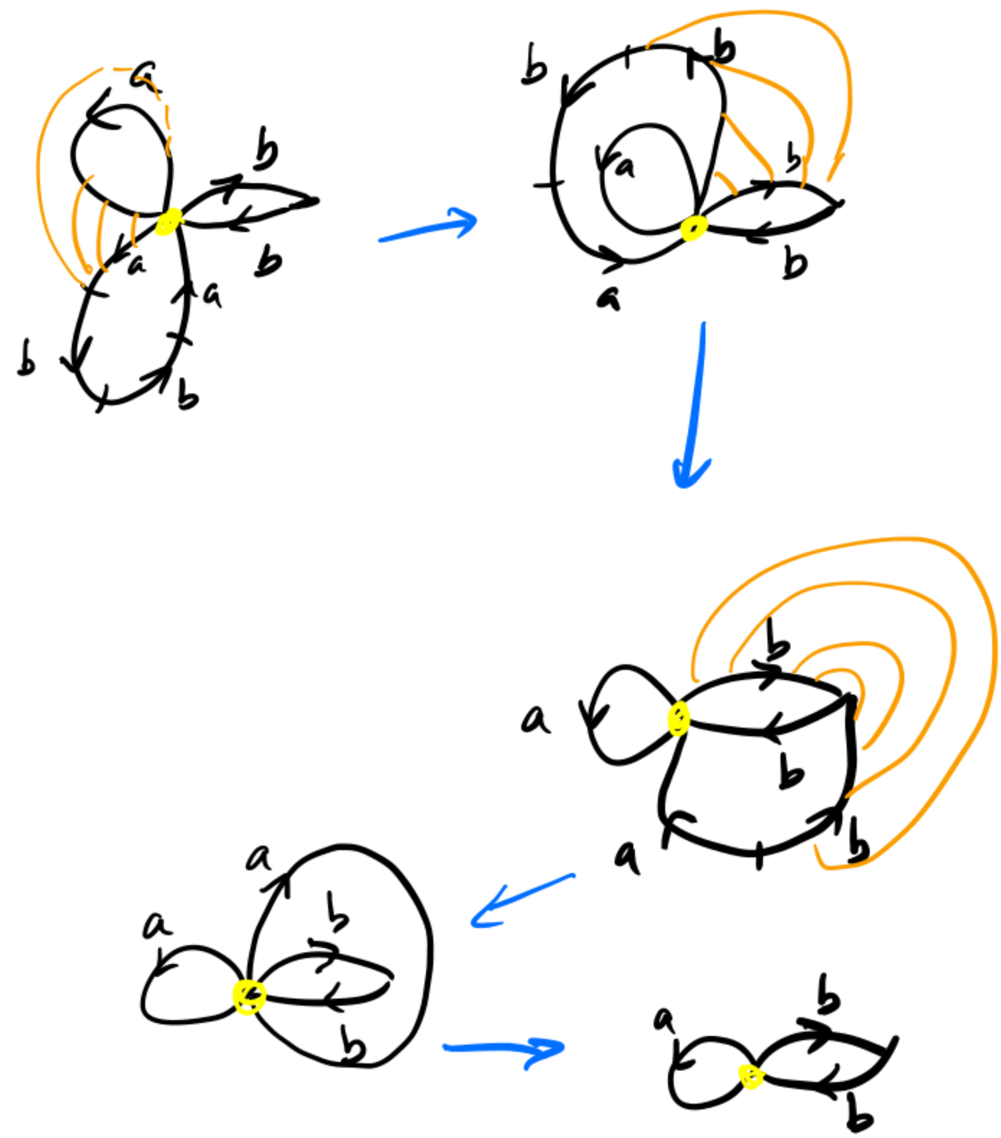
Given a subgroup $H < F_n$ generated by an explicit finite set, we start with the rose whose petals are labeled by these generators. This defines a morphism to the rose R representing F_n, and we fold until we get an immersion $\Gamma_H \looparrowright R$.

Example 1. $H = \langle ab, a^2, b^2 \rangle$.



Example 2. $H = \langle a, abba, bb \rangle$.

$$H = \langle a, abba, bb \rangle$$



Here are the basic questions we can answer:

(1) Find a basis of H. This is done by choosing loops in Γ_H that represent a basis for $\pi_1(\Gamma_H)$ and reading off labels. In Example 1, the given generators form a basis, and in Example 2 $\{a, bb\}$ is a basis. This procedure proves that finitely generated subgroups of free groups are free. This is true even for subgroups that are not finitely generated.

(2) Given $w \in F_n$ decide if $w \in H$. If w is represented by a reduced word, the answer is "yes" if and only if w can be lifted to an edge path in Γ_H that starts and ends at the basepoint. In Example 1, $\overline{a}b \in H$ but $\overline{a}bb \notin H$.

(3) Given $w \in F_n$ decide if w is conjugate to an element in H. The answer is "yes" if and only if the reduced word w can be lifted to a path in Γ_H that starts and ends at the same point v (not necessarily basepoint; a conjugating element is represented by a path joining the basepoint with this point v). In Example 2, $bab \notin H$ but it is conjugate into H, e.g. $\overline{b}(bab)b = abb \in H$.

(4) Compute the index $[F_n : H]$ of H in F_n. If the immersion $\Gamma_H \looparrowright R$ is a covering map (necessarily finite-sheeted) the index is finite and equal to the number of sheets. Otherwise $\Gamma_H \looparrowright R$ can be completed to a covering map (with infinitely many sheets) by attaching trees and the index is then infinite. Either way, one can give an explicit set of coset representatives. In Example 1 the index is 2, and in Example 2 it is infinite. Coset representatives in Example 1 are 1 and a (for example).

Problems about homomorphisms between free groups. If $\phi : F_n \to F_m$ is a homomorphism, decide if ϕ is injective, surjective, bijective.

Represent ϕ by a labeled rose representing F_n where the labels come from F_m. Perform the folding procedure to arrive at $\Gamma \looparrowright R$.

- ϕ is injective if and only if all folds are of type 1.
- ϕ is surjective if and only if $\Gamma \looparrowright R$ is a homeomorphism.
- Of course, ϕ is an isomorphism if and only if it is both injective and surjective.

The two examples above can be interpreted as homomorphisms. In Example 1 we have

$$F_3 = \langle x, y, z \rangle \to F_2 = \langle a, b \rangle$$

given by $x \mapsto ab, y \mapsto a^2, z \mapsto b^2$. This homomorphism is injective but not surjective.

In Example 2 we have $F_3 = \langle x, y, z \rangle \to F_2 = \langle a, b \rangle$ given by $x \mapsto a, y \mapsto abba, z \mapsto b^2$. This homomorphism is neither injective nor surjective.

We'll also learn how to find the intersection of two subgroups. See the Notes.