Math 5510 Homework 6

Klein bottle

1. Let Γ be a free and properly discontinuous group of isometries of $\mathbb{R}^2$ and let $\alpha, \beta \in \Gamma$ be two glide reflections. Show that the axes of α, β are parallel to each other. (Hint: Show that otherwise $\alpha\beta$ is a rotation.)

2. Let Γ be a free and properly discontinuous group of isometries of $\mathbb{R}^2$ and let $\alpha, \beta \in \Gamma$ be such that α is a glide reflection with axis ℓ and β is a translation by the vector $v \in \mathbb{R}^2$. Show that either Γ contains a glide reflection with the same axis ℓ but with smaller translation length, or the vector v projected to ℓ has length an integral multiple of the translation length of α.

Torus

3. Use your knowledge of the moduli space of flat tori to prove that if $X = \mathbb{R}^2/\Gamma$ is a flat torus and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by angle $\neq 0, \pi$ such that $F\Gamma F^{-1} = \Gamma$ (i.e. F induces an isometry of X) then X is a square torus or a hexagon torus. (Hint: Show that the other tori in the moduli space do not admit such a rotation, by considering lengths of closed geodesics.)

4. Let X be the square torus of area 1 and consider the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Show that L induces a homeomorphism $h : X \to X$ and find all parallel families of geodesics preserved by h. (Hint: There are two. Individual geodesics in these families may not be preserved, but they should be sent to parallel geodesics.)