Math 5510 Homework 5

Recall that an orientation preserving isometry of $\mathbb{R}^2 = \mathbb{C}$ is given analytically by

$$z \mapsto az + b$$

with $a, b \in \mathbb{C}$ and $|a| = 1$. Likewise, an orientation reversing isometry is given by

$$z \mapsto a\overline{z} + b$$

with $|a| = 1$. Also recall that every non-identity isometry is one of translation, rotation, reflection, glide reflection.

1. Every nonidentity isometry satisfies one of the following:
   (a) it has a line of fixed points, or
   (b) it has a single fixed point, or
   (c) it has no fixed points and a parallel family of invariant lines, or
   (d) it has no fixed points and a single invariant line.

2. (a) Show that if two bijections $f, g : X \to X$ of a set X commute (i.e. $fg = gf$) then f preserves the set of fixed points of g and vice versa (this means that if x is a fixed point of g then so is $f(x)$).
   (b) Prove that if two non-identity orientation preserving isometries of $\mathbb{R}^2$ commute then they are both translations or they are both rotations about the same point.

3. Find the axis of the orientation reversing isometry $z \mapsto a\overline{z} + b$ (the axis of a reflections is the fixed line, and of a glide reflection it is the unique invariant line).

4. Prove that a glide reflection is not the composition of one or two reflections.