**Question 1.** (Artin 2.8.6) Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18$, $|G'| = 15$, and that $\varphi$ is not the trivial homomorphism. What is the order of the kernel?

**Solution.** Let $k = |\ker \varphi|$. Note $\ker \varphi \leq G$ so $k|18$. From the first isomorphism theorem, we also have $G/\ker \varphi \cong \varphi(G) \leq G'$, so $\frac{18}{k}|15$. Combining those, the only possible values for $k$ are 6, 18. However, since $\varphi$ is assumed to be nontrivial, $k \neq 18$ is not valid. Hence, $k = |\ker \varphi| = 6$. //

**Question 2.** (Artin 2.8.9) Let $G$ be a finite group. Under what circumstances is the map $\varphi : G \to G$ defined by $\varphi(x) = x^2$ an automorphism of $G$?

**Solution.** We claim that $G$ is abelian and has odd order. First, as $\varphi$ is a homomorphism we see that for every $a, b \in G$:

$$a^2b^2 = \varphi(a)\varphi(b) = \varphi(ab) = (ab)^2 = abab,$$

which implies that $ab = ba$, so $G$ should be abelian.

On the other hand, for $\varphi$ to be injective, every element has to have odd order. Indeed, say $a \in G$ has order $2m$ for some positive integer $m$. Then by definition of order, $a^m \neq 1$ and $a^m \in G$ but $\varphi(a^m) = a^{2m} = 1$ breaking the injectivity of $\varphi$. Hence, every element of $G$ has odd order. This implies the order of $G$ is odd. (Contrapositive of “Group of even order has order 2 element.”) //

**Question 3.** (Artin 2.11.4) In each of the following cases, determine whether or not $G$ is isomorphic to the product group $H \times K$.

(a) $G = \mathbb{R}^\times$, $H = \{\pm 1\}$, $K = \{$positive real numbers$\}$.

(b) $G = \{$invertible upper triangular $2 \times 2$ matrices$\}$, $H = \{$invertible diagonal matrices$\}$, $K = \{$upper triangular matrices with diagonal entries 1$\}$.

(c) $G = \mathbb{C}^\times$, $H = \{$unit circle$\}$, $K = \{$positive real numbers$\}$.

**Solution.** (a) **Yes**, $G \cong H \times K$. The map is $r \mapsto (\frac{r}{|r|}, |r|)$.

(b) **No**, $G \not\cong H \times K$. Although the map $(D, U) \mapsto D \cdot U$ gives $G = HK$, one can check that $H$ is not normal in $G$ (that is, a conjugate of a diagonal matrix may not be diagonal.), so fails to be direct product.

(c) **Yes**, $G \cong H \times K$. The map is $x \mapsto (|x|, e^{i \text{arg}(x)})$ where $|x|$ is the magnitude of $x$ and $\text{arg}(x)$ is the argument of $x$. //
**Question 4** (Artin 2.12.2). In the general linear group $GL_3(\mathbb{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $*$ represents an arbitrary real number. Show that $H$ is a subgroup of $GL_3(\mathbb{R})$, that $K$ is a normal subgroup of $H$, and identify the quotient group $H/K$. Determine the center of $H$.

**Proof.** Here we sketch how do we find the quotient group $H/K$. The key is to come up with a surjective homomorphism from $H$, with kernel $K$. One natural construction of such homomorphism is:

$$\varphi : H \longrightarrow (\mathbb{R}^2, +), \quad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \longmapsto (a, c) \in \mathbb{R}^2.$$

Here $(\mathbb{R}^2, +)$ is the group $\mathbb{R} \times \mathbb{R}$ equipped with componentwise addition. One can check that this is a surjective homomorphism. Hence, $H/K \cong \varphi(H) \cong (\mathbb{R}^2, +)$.

To find the center of $H$, one can observe from computation that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if and only if} \quad ac' = a'c.$$

Hence, to commute with an arbitrary element $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in H$, it follows that $a' = c' = 0$. Therefore $K \subset Z(H)$. The other containment $K \supset Z(H)$ can be verified by direct computation. □

**Question 5** (Artin 2.12.4). Let $H = \{ \pm 1, \pm i \}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth root of unity. Describe the cosets of $H$ in $G$ explicitly. Is $G/H$ isomorphic to $G$?

**Solution.** Recall $aH = bH$ if and only if $ab^{-1} \in H$. Hence we can describe the cosets of $H$ as:

$$aH = \{ \pm a, \pm ai \} = -aH = iaH = -iaH$$

for $a \in G = \mathbb{C}^\times$.

We can realize $H$ as the kernel of the fourth power homomorphism:

$$\varphi : G \longrightarrow G, \quad z = re^{i\theta} \longmapsto z^4 = r^4 e^{4i\theta},$$

where $r$ is a positive real number and $\theta \in [0, 2\pi)$. Since $r^4 e^{4i\theta}$ still represents every nonzero complex number, $\varphi$ is surjective. Therefore, the quotient $G/H$ is still isomorphic to $G$. //

**Question 6** (Artin 2.M.14). Prove that the two matrices

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ E' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

generate the group $SL_2(\mathbb{Z})$ of all integer matrices with determinant 1.

**Proof.** The key is the Euclidean algorithm. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Since $ad - bc = 1$, we have $\gcd(a, c) = 1$. Note $E$ and $E'$ each represents a row operation $R_1 \rightarrow R_1 + R_2$ and $R_2 \rightarrow R_1 + R_2$. Hence, using powers of $E, E'$ one can perform Euclidean algorithm on the
pair \((a, c)\) to make it \((1, 0)\) or \((0, 1)\). After reducing \((a, c)\) to \((1, 0)\) or \((0, 1)\), there are only limited choices for \(b, d\), so the resulting matrix will be one of the forms:

\[
\begin{bmatrix}
1 & m \\
0 & 1 \\
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
0 & -1 \\
1 & n \\
\end{bmatrix},
\]

for some \(m, n \in \mathbb{Z}\). Note that \(\begin{bmatrix}
1 & m \\
0 & 1 \\
\end{bmatrix} = E^m\), and a matrix of the second form can be reduced to the first form by row operations:

\[
\begin{bmatrix}
0 & -1 \\
1 & n \\
\end{bmatrix} \xrightarrow{R_1 \mapsto R_1 + R_2} \begin{bmatrix}
1 & n - 1 \\
1 & n \\
\end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - R_1} \begin{bmatrix}
1 & n - 1 \\
0 & 1 \\
\end{bmatrix},
\]

which implies that \(E'^{-1} E \begin{bmatrix}
0 & -1 \\
1 & n \\
\end{bmatrix} = \begin{bmatrix}
1 & n - 1 \\
0 & 1 \\
\end{bmatrix}\).

Therefore, any matrix in \(SL_2(\mathbb{Z})\) can be row-reduced using \(E, E'\) to one of the forms \(\begin{bmatrix}
1 & m \\
0 & 1 \\
\end{bmatrix}, \begin{bmatrix}
0 & -1 \\
1 & n \\
\end{bmatrix}\), which can be further expressed as a product of \(E, E'\) and their inverses. This concludes that \(SL_2(\mathbb{Z}) = \langle E, E' \rangle\). \(\square\)