**Question 1** (Artin 2.5.1). Let $\varphi : G \to G'$ be a surjective homomorphism. Prove that if $G$ is cyclic, then $G'$ is cyclic, and if $G$ is abelian, then $G'$ is abelian.

**Proof.** Suppose $G$ is cyclic, and let $G = \langle g \rangle$ for some $g \in G$. Then $\varphi(g^k) = \varphi(g)^k$ for every $k \in \mathbb{Z}$, so $\varphi(G) = \langle \varphi(g) \rangle$. Because $\varphi$ is surjective, it follows that $G' = \langle \varphi(g) \rangle$ proving $G'$ is cyclic.

Now suppose $G$ is abelian. Pick arbitrary elements $g', h' \in G'$, and then there exist $g, h \in G$ such that $\varphi(g) = g'$ and $\varphi(h) = h'$ by surjectivity. Then

$$g'h' = \varphi(g)\varphi(h) = \varphi(gh) = \varphi(hg) = \varphi(h)\varphi(g) = h'g',$$

proving $G'$ is abelian. □

**Question 2** (Artin 2.5.3). Let $U$ denote the group of invertible triangular $2 \times 2$ matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and let $\varphi : U \to \mathbb{R}^\times$ be the map that sends $A \mapsto a^2$. Prove that $\varphi$ is a homomorphism, and determine its kernel and image.

**Proof.** Note $a \cdot a' = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bd' \\ 0 & dd' \end{bmatrix}$, so

$$\varphi \left( \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right) \varphi \left( \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix} \right) = a^2 a'^2 = \varphi \left( \begin{bmatrix} a a' & ab' + bd' \\ 0 & dd' \end{bmatrix} \right),$$

proving $\varphi$ is a homomorphism.

The kernel of $\varphi$ is the set of invertible triangular matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $a^2 = 1$, so $a = \pm 1$. Since it is invertible, $d$ should be nonzero. Hence,

$$\ker \varphi = \left\{ \begin{bmatrix} \pm 1 & b \\ 0 & d \end{bmatrix} \bigg| b, d \in \mathbb{R}, \text{ and } d \neq 0 \right\}.$$

For the image of $\varphi$, note that the entry $a$ of $A$ can be any nonzero real number, so $a^2$ can be all positive real. Hence, $\varphi(U) = \mathbb{R}_{>0}$, the set of all positive real numbers. □
**Question 3** (Artin 2.5.6). Determine the center of \( GL_n(\mathbb{R}) \). (The homework was to prove this when \( n = 2 \).)

**Solution.** When \( n = 2 \), you could prove the center is the set \( \{ kI_2 \mid k \in \mathbb{R} \} \) where \( I_2 \) is the \( 2 \times 2 \) matrix by computing matrix multiplication. Here we give a sketch the idea for general \( n \geq 2 \). Let \( C \in Z(GL_n(\mathbb{R})) \) be a center element. As given in the hint, we use the fact that \( C \) commutes with elementary matrices. Note the elementary matrices are multiplied from the left, they give elementary row operations but multiplied from the right, they give elementary column operations. Using this, we can narrow down the form of \( C \) in the following order:

- By commuting \( C \) with an elementary matrix of the first kind (add a multiple of one row/column to another), one can conclude \( C \) should be a diagonal matrix, i.e. all the off diagonal entries are zero.
- By commuting \( C \) with an elementary matrix of the third kind (multiply a row/column by a nonzero constant), one can conclude all the diagonal entries of \( C \) should be the same.

Therefore, we can conclude that \( Z(GL_n(\mathbb{R})) \subset \{ kI_n \mid k \in \mathbb{R} \setminus \{0\} \} \) where \( I_n \) is the \( n \times n \) identity matrix. The converse is also true, that all the matrices of the form \( kI_n \) are in the center of \( GL_n(\mathbb{R}) \) because multiplying by \( kI_n \) (from either left or right) is just a scalar multiplication by \( k \). This proves that the center is indeed \( \{ kI_n \mid k \in \mathbb{R} \setminus \{0\} \} \).

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**Question 4** (Artin 2.6.1). Let \( G' \) be the group of real matrices of the form \( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \). Is the map \( \mathbb{R}^+ \rightarrow G' \) that sends \( x \) to this matrix an isomorphism?

**Proof.** Yes, it is an isomorphism. Note \( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} \), so \( G' \) is a multiplicative group. Hence, the map \( \varphi : \mathbb{R}^+ \rightarrow G' \) defined as \( \varphi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \) is a homomorphism:

\[
\varphi(x + y) = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \varphi(x)\varphi(y).
\]

Also \( \varphi \) is injective, as \( \ker \varphi = 0 \), where the identity element of \( G' \) is the identity matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Finally, \( \varphi \) is surjective by definition of \( G' \). This concludes that \( \varphi \) is an isomorphism. \( \square \)

**Question 5** (Artin 2.6.8). Prove that \( A \mapsto (A^t)^{-1} \) is an automorphism of \( GL_n(\mathbb{R}) \).

**Proof.** First, observe that transposing and taking inverses are commutative on \( GL_n(\mathbb{R}) \). That is, \( (A^t)^{-1} = (A^{-1})^t \). To see why, note for \( A \in GL_n(\mathbb{R}) \):

\[
A^t \cdot (A^{-1})^t = (A^{-1} \cdot A)^t = I_n^t = I_n,
\]

where \( I_n \) is the \( n \times n \) identity matrix. Now we claim that the map \( \varphi : GL_n(\mathbb{R}) \) defined as \( \varphi(A) = (A^t)^{-1} \) is a homomorphism, and it has the inverse as its own, which implies that \( \varphi \) is an automorphism.

First, note

\[
\varphi(AB) = ((AB)^t)^{-1} = (B^tA^t)^{-1} = (A^t)^{-1}(B^t)^{-1} = \varphi(A)\varphi(B),
\]

so \( \varphi \) is a homomorphism. To show \( \varphi^{-1} = \varphi \), we prove \( \varphi \circ \varphi = \text{Id} \):

\[
(\varphi \circ \varphi)(A) = \varphi(((A^t)^{-1})^t) = (((A^t)^{-1})^t)^{-1} = \text{Id},
\]

where we used the earlier observation that transposing and taking inverses are commuting. All in all, \( \varphi \) is a homomorphism whose inverse, the same as \( \varphi \), is also a homomorphism. This concludes that \( \varphi \) is an automorphism. \( \square \)
**Question 6 (Bonus question: Artin 2.6.3).** Show that the functions $f = 1/x$ and $g = (x-1)/x$ generate a group of functions, the law of composition being composition of functions, that is isomorphic to the symmetric group $S_3$.

**Proof.** There are multiple ways to prove this. One way to prove this is that, first observe $f$ has order 2 and $g$ has order 3. Then consider a map that sends $f$ to a transposition $(1, 2) \in S_3$ and sends $g$ to a permutation $(1, 2, 3) \in S_3$. One can check that this map is a homomorphism. It is surjective as $(1, 2)$ and $(1, 2, 3)$ generate the whole $S_3$, and then it is automatically injective by noting that the order of $\langle f, g \rangle$ is the same as that of $S_3$, which is 6.

Here we sketch another idea of proof by analyzing how the group acts on a six point set $X = \{\frac{1}{3}, 3, \frac{2}{3}, \frac{2}{3}, -\frac{1}{2}, -2\}$. Start with $3 \in \mathbb{R}$. After applying $f$ and $g$ to $3$ multiple times, one can notice that those multiplications end up six real numbers in $X$ as in the following figure. Note every vertex has valence four; incoming $f, g$ and outgoing $f, g$.

**Figure 1.** Graphical representation of the group $\langle f, g \rangle \cong D_6 \cong S_3$.

Note that this diagram has the same “shape” if you replace each number with properly labeled triangle, where the resulting diagram represents how dihedral group $D_6$ of order 6 flips and rotates a triangle. Hence, forgetting the role of vertices and edges of each diagram, those two diagrams are completely identical; which implies that the group structures of $\langle f, g \rangle$ and $D_6$ are identical, so we can conclude $\langle f, g \rangle \cong D_6$, where we know $D_6 \cong S_3$ finishing the proof. If you want to learn more “shapes” of those groups, see e.g. the Wikipedia article of *Cayley Graph*. \qed